About Rogers semilattices of finite families in Ershov hierarchy

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Let $S$ be countable set. Any surjective map from $\omega$ onto $S$ we will call a *numbering*.

Goncharov-Sorbi approach:

Numbering $\eta$ is called $\Gamma$-computable, if set $\{< x, y > | y \in \eta_x \} \in \Gamma$
$Com_\Gamma(S)$ – family of all $\Gamma$-computable numberings of $S$.

$\mu \leq \nu$ if there is computable function $f$ and $\mu(x) = \nu(f(x))$.

$\langle Com_\Gamma(S)/\equiv, \leq \rangle$ – Rogers semilattice $R_\Gamma(S)$.

Any member of a greatest element of $R_\Gamma(S)$ we will call a principal $\Gamma$-computable numbering.
A is $\Sigma_n^{-1}$-set, if $A(x) = \lim_{s} A(x, s)$, $A(x, 0) = 0$ and $|\{s|A(x, s) \neq A(x, s + 1)\}| \leq n$
Let $\mathcal{O}$ be Kleene ordinal notation system, $A \subseteq \omega$ and $a$ is notation for ordinal $\alpha$ in $\mathcal{O}$.

For all $a \in \mathcal{O}$ set $A$ is $\Sigma_{a}^{-1}$-set if there exist total computable function $f(x, s)$ and partial computable function $g(x, s)$ and for all $x \in \omega$:

1. $A(x) = \lim_{s} f(x, s)$, $f(x, 0) = 0$;
2. $g(x, s) \downarrow \rightarrow g(x, s + 1) \downarrow \leq_{o} g(x, s) <_{o} a$;
3. $f(x, s) \neq f(x, s + 1) \rightarrow g(x, s + 1) \downarrow \neq g(x, s)$. 
Theorem (Herbert, Jain, Lempp, Mustafa, Stephan)
There is an operator $\mathcal{E}$ that for any $\Sigma^{-1}_n$-computable family $S$, $\mathcal{E}(S)$ is $\Sigma^{-1}_{n+1}$-computable family, and $\mathcal{R}^{-1}_n(S)$ is isomorphic to $\mathcal{R}^{-1}_{n+1}(\mathcal{E}(S))$
Theorem (Lachlan)
Any finite family of c.e. sets has a computable principal numbering.

Theorem (Badaev, Goncharov, Sorbi)
Let $S$ be any finite $\Sigma^0_{n+2}$-computable family of sets. $S$ has a $\Sigma^0_{n+2}$-computable principal numbering if and only if there is least set under inclusion in $S$. 
**Theorem (Abeshev)**

There is a family $S = \{A, B\}$ of disjoint $\Sigma_2^{-1}$-sets without $\Sigma_2^{-1}$-computable principal numbering.
Proposition
For any ordinal notation $\alpha > \omega 2$, any finite family of effective disjoint $\Sigma_2^{-1}$-sets has a $\Sigma_\alpha^{-1}$-computable principal numbering.

Here sets are effective disjoint, when sets $\{x|\exists s f(x, s) = 1\}$ are disjoint.
Theorem (Bazhenov, Mustafa, O.)

For any ordinal notation $\alpha$ of a non-zero ordinal, any family $S = \{A, B\}$ of c.e. sets has a $\Sigma^{-1}_\alpha$-computable principal numbering.
Proposition (Bazhenov, Mustafa, O.)
Let $S = \{A, B\}$ be a family of c.e. sets with $A \subset B$, $B \setminus A$ is not c.e. Then any $\Sigma_{2n+2}$-computable numbering of $S$ is equivalent to some $\Sigma_{2n+1}$-computable numbering of $S$.

The same goes for family $\{\emptyset, B \setminus A\}$ and the levels $2n$ and $2n + 1$. 
Lemma (Lachlan)
Family $S$ of c.e. sets has a computable principal numbering if and only if $S \setminus \{\emptyset\}$ has one too.
Let $\mathcal{P} = \langle P, \leq_P \rangle$ be a finite partially ordered set. Let

$\tilde{p} = \{ x | p \leq_P x \}$. We will call a family $\{ F_p \}_{p \in \mathcal{P}}$ of nonempty $\Sigma_a^{-1}$-sets acceptable if $F_{p_1} \cap F_{p_2} = \bigcup_{q \in \tilde{p}_1 \cap \tilde{p}_2} F_q$ for any $p_1, p_2 \in \mathcal{P}$.

**Theorem (Bazhenov, Mustafa, O.)**

Let $a$ be the ordinal notation of nonzero ordinal. For any finite partially ordered set $\mathcal{P}$ and any acceptable family $\{ F_p \}_{p \in \mathcal{P}}$, there is $\Sigma_a^{-1}$-computable principal numbering of family $\{ F_p \}_{p \in \mathcal{P}} \cup \{ \emptyset \}$.
Corollary
Let $S$ be a finite family of disjoint $\Sigma_a^{-1}$-sets, there is $\Sigma_a^{-1}$-computable principal numbering of family $S \cup \{\emptyset\}$

Corollary
Let $S = \{\emptyset \subset A_1 \subset \cdots \subset A_n\}$ be a finite family of $\Sigma_a^{-1}$-sets, then there is $\Sigma_a^{-1}$-computable principal numbering of $S$
Thanks for your attention!