About Rogers semilattices of finite families in Ershov hierarchy

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Let $\mathcal{S}$ be countable set. Any surjective map from ω onto $\mathcal{S}$ we will call a *numbering*.

Goncharov-Sorbi approach:

Numbering η is called Γ-computable, if set $\{< x, y > | y \in \eta_x\} \in \Gamma$

$Com_\Gamma(\mathcal{S})$ – family of all Γ-computable numberings of S.

$\mu \leqslant \nu$ if there is computable function f and $\mu(x) = \nu(f(x))$.

$\langle Com_\Gamma(\mathcal{S})/_\equiv, \leqslant\rangle$ – Rogers semilattice $\mathcal{R}_\Gamma(\mathcal{S})$.

Any member of a greatest element of $\mathcal{R}_\Gamma(\mathcal{S})$ we will call a *principal* Γ-computable numbering.

A is Σ_n^{-1}-set, if $A(x) = \lim_s A(x,s)$, $A(x,0) = 0$ and $|\{s | A(x,s) \neq A(x,s+1)\}| \leq n$

Let $\mathcal{O}$ be Kleene ordinal notation system, $A \subseteq \omega$ and a is notation for ordinal α in $\mathcal{O}$.

For all $a \in O$ set A is Σ_a^{-1}-set if there exist total computable function $f(x,s)$ and partial computable function $g(x,s)$ and for all $x \in \omega$:

1. $A(x) = \lim_s f(x,s),\ f(x,0) = 0$;
2. $g(x,s) \downarrow \rightarrow g(x,s+1) \downarrow \leq_o g(x,s) <_o a$;
3. $f(x,s) \neq f(x,s+1) \rightarrow g(x,s+1) \downarrow \neq g(x,s)$.

Theorem (Herbert, Jain, Lempp, Mustafa, Stephan)
There is an operator $\mathcal{E}$ that for any Σ_n^{-1}-computable family $\mathcal{S}$, $\mathcal{E}(\mathcal{S})$ is Σ_{n+1}^{-1}-computable family, and $\mathcal{R}_n^{-1}(\mathcal{S})$ is isomorphic to $\mathcal{R}_{n+1}^{-1}(\mathcal{E}(\mathcal{S}))$

Theorem (Lachlan)
Any finite family of c.e. sets has a computable principal numbering.

Theorem (Badaev,Goncharov, Sorbi)
Let $\mathcal{S}$ be any finite Σ^0_{n+2}-computable family of sets. $\mathcal{S}$ has a Σ^0_{n+2}-computable principal numbering if and only if there is least set under inclusion in $\mathcal{S}$.

Theorem (Abeshev)
There is a family $\mathcal{S} = \{A, B\}$ *of disjoint* Σ_2^{-1}*-sets without* Σ_2^{-1}*-computable principal numbering.*

Proposition
For any ordinal notation $a >_{\mathcal{O}} 2$, *any finite family of effective disjoint* Σ_2^{-1}*-sets has a* Σ_a^{-1}*-computable principal numbering.*

Here sets are effective disjoint, when sets $\{x | \exists s f(x,s) = 1\}$ are disjoint.

Theorem (Bazhenov, Mustafa, O.)
For any ordinal notation a of a non-zero ordinal, any family $\mathcal{S} = \{A, B\}$ of c.e. sets has a Σ^{-1}_a-computable principal numbering.

Proposition (Bazhenov, Mustafa, O.)
Let $\mathcal{S} = \{A, B\}$ be a family of c.e. sets with $A \subset B$, $B \setminus A$ is not c.e. Then any Σ^{-1}_{2n+2}-computable numbering of $\mathcal{S}$ is equivalent to some Σ^{-1}_{2n+1}-computable numbering of $\mathcal{S}$.

The same goes for family $\{\emptyset, B \setminus A\}$ and the levels $2n$ and $2n+1$.

Lemma (Lachlan)
Family $\mathcal{S}$ *of c.e. sets has a computable principal numbering if and only if* $\mathcal{S} \setminus \{\emptyset\}$ *has one too.*

Let $\mathcal{P} = \langle P, \leq_P \rangle$ be a finite partially ordered set. Let $\check{p} = \{x | p \leq_P x\}$. We will call a family $\{F_p\}_{p \in \mathcal{P}}$ of non empty Σ_a^{-1}-sets *acceptable* if $F_{p_1} \bigcap F_{p_2} = \bigcup\limits_{q \in \check{p}_1 \cap \check{p}_2} F_q$ for any $p_1, p_2 \in \mathcal{P}$.

Theorem (Bazhenov, Mustafa, O.)
Let a be the ordinal notation of nonzero ordinal. For any finite partially ordered set $\mathcal{P}$ and any acceptable family $\{F_p\}_{p \in \mathcal{P}}$, there is Σ_a^{-1}-computable principal numbering of family $\{F_p\}_{p \in \mathcal{P}} \bigcup \{\emptyset\}$

Corollary
Let $\mathcal{S}$ be a finite family of disjoint Σ_a^{-1}-sets, there is Σ_a^{-1}-computable principal numbering of family $\mathcal{S}\bigcup\{\emptyset\}$

Corollary
Let $\mathcal{S} = \{\emptyset \subset A_1 \subset \cdots \subset A_n\}$ be a finite family of Σ_a^{-1}-sets, then there is Σ_a^{-1}-computable principal numbering of $\mathcal{S}$

Thanks for your attention!

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