A Hyperintensional and Paraconsistent Approach to Belief Dynamics

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Suppose our agent is the investigator of the missing pearls. She believes that the maid has the key to the safe of the pearls (p) and the butler has the key to the safe (q). During the questioning, she learns that the maid indeed does not have a key ($\neg p$).
We can model her initial information and belief state, how her information state is updated with the new information and how her beliefs are revised.

The dynamic belief base models represent reasoning on two levels.

- First is on the level of information and how agents are able to combine different collections of information.
- Second is on the level of belief and how consistent belief sets are formed based on possibly inconsistent and incomplete information.
- How the information states and beliefs are revised with new information.

A dynamic belief base model represents the doxastic state of a single agent.

Outline

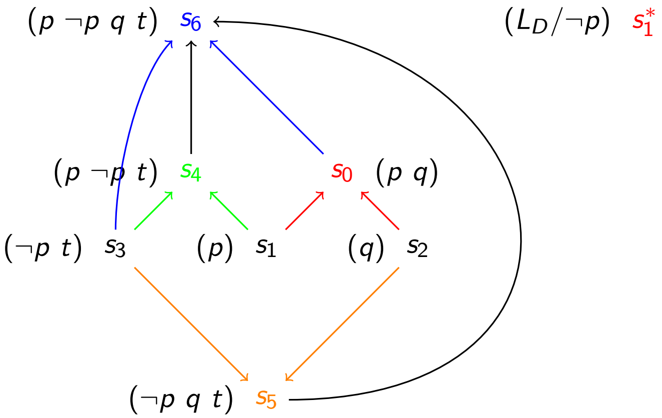
- The static belief base model
- The dynamic belief base model
- Results & Conclusion

Language

- AT $:= p_1, p_2, p_3 ...$
- $\phi ::=$ AT
  $\neg\phi \mid \phi \wedge \psi \mid \phi \vee \psi \mid \phi \rightarrow \psi \mid B\phi \mid [\phi]_G\psi \mid \langle\phi\rangle_G\psi \mid [\phi]_L\psi \mid \langle\phi\rangle_L\psi$
- $\top$ (always true) and $\bot$ (always false)
- The set of all formulas is denoted by L_D, with the usual subsets for propositional language L_{prop} and literals $l \in L_D$.

- **The static belief base model**
- The dynamic belief base model
- Results & Conclusion

$M = \langle S, V, \circ, \perp, \leq \rangle$



The Static Belief Base Model - Terminology

Given a static belief base model M on the situations space S,

- The fusion function creates a **partial order on** S: $s' \sqsubseteq s$ iff $s \circ s' = s$.
- **A situation** $s \in S$ **is consistent** iff $s \not\leq s$. Otherwise it is inconsistent.
- **Best sets of situations in a set** $I \subseteq S$ is denoted by $min_{\leq_M}(I) = \{A \subseteq I | \forall\ B \subseteq I,\ A \leq B\}$.

The Static Belief Base Model - Terminology

- **A set of situations** $A \subseteq S$ **is consistent** iff for all $s, s' \in A$, $s \not\perp s'$. Otherwise it is inconsistent.
- A set of situations $A \subseteq S$ is **maximally consistent with respect to a situation** $s \in S$ iff A is consistent and for all $s' \in S$ if $s' \sqsubseteq s$ and $s' \notin A$ then $A \cup \{s'\}$ is inconsistent.
- **Best of a situations** s is such that $Best_M(s) = min_{\leq_M}(\{A \subseteq S \mid A$ is a maximally consistent with respect to $s\})$.

Satisfaction clauses for (some) formulas in L_D are as follows:

$s \models v$ iff $v \in V(s)$

$s \models \neg v$ iff $\neg v \in V(s)$

$s \models \neg\phi$ iff for all s', if $s' \models \phi$ then $s \bot s'$ (iff $s^* \not\models \phi$)

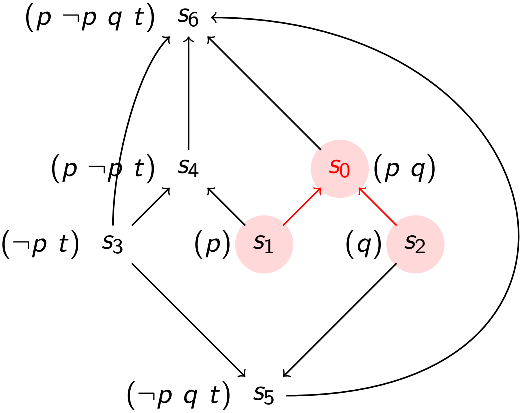
$s \models \phi \wedge \psi$ iff $s \models \phi$ and $s \models \psi$

$s \models \phi \vee \psi$ iff $s \models \phi$ or $s \models \psi$

$s \models \phi \to \psi$ iff for all s', if $s \sqsubseteq s'$ and $s' \models \phi$ then $s' \models \psi$

$s \models B\phi$ iff for all $A \in Best(s)$, for some $s' \in A$, $s' \models \phi$.

$s \models \top$



(L_D/p) s_1^*

For all $A, B \subseteq S$, $A \leq_M B$

$Best_M(s_0) = \{\{s_0, s_1, s_2\}\}$
$(M, s_0) \models B(p \wedge q)$

- The static belief base model
- **The dynamic belief base model**
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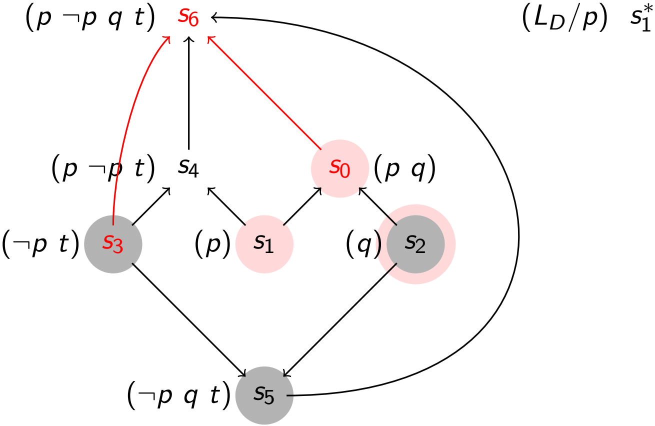
$M^+ = \langle S, V, \circ, \perp, \leq, [\![R\cdot]\!]_G, [\![R\cdot]\!]_L \rangle$



For all $A, B \subseteq S$, $A \leq_{M^*} B$ iff $A \cap [\![\neg p]\!] \neq \emptyset$ and $B \cap [\![\neg p]\!] = \emptyset$
$Best_{M^*}(s_6) = \{\{s_2, s_3, s_5\}\}$
$(M^*, s_6) \models B(\neg p \wedge q) \wedge Bt$

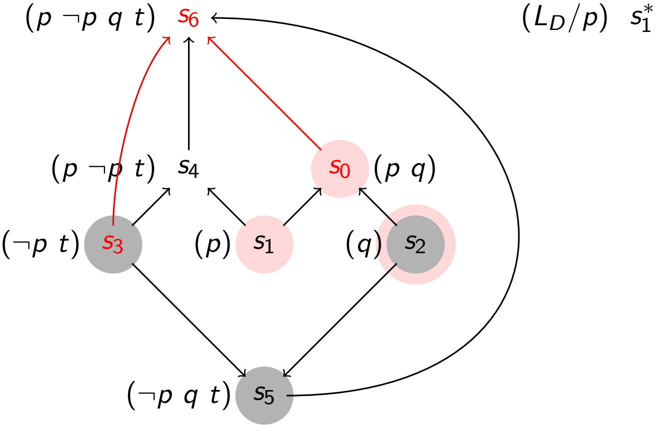
Satisfaction clauses for (some) formulas in L_D are as follows:

- $(M, s) \models [\phi]_{G/L}\psi$ iff

  $\forall (M', s') : (M, s)[\![R\phi]\!]_{G/L}(M', s') \implies (M', s') \models \psi.$

- $(M, s) \models \langle\phi\rangle_{G/L}\psi$ iff

  $\exists (M', s') : (M, s)[\![R\phi]\!]_{G/L}(M', s')$ and $(M', s') \models \psi.$



$(M^*, s_6) \models B(\neg p \wedge q) \wedge Bt$
$(M, s_0) \models B(p \wedge q) \wedge [\neg p]_G B(\neg p \wedge (q \wedge t))$

- The static belief base model
- The dynamic belief base model
- **Results** & **Conclusion**

$\vdash \top$

$\vdash A \rightarrow A$

$\vdash A \rightarrow A \vee B$

$\vdash B \rightarrow A \vee B$

$\vdash A \wedge (B \vee C) \leftrightarrow (A \wedge B) \vee (A \wedge C)$

$\vdash A \vee (B \wedge C) \leftrightarrow (A \vee B) \wedge (A \vee C)$

$\vdash A \leftrightarrow \neg\neg A$

$\vdash \neg A \vee \neg B \leftrightarrow \neg(A \wedge B)$

$\vdash \neg A \wedge \neg B \leftrightarrow \neg(A \vee B)$

$A, A \rightarrow B \vdash B$ (Modus Ponens)

Axioms for L_{prop}

$\vdash A \to (B \to A)$

$\vdash A \to (B \to A \wedge B)$

$\vdash A \wedge B \to A$

$\vdash A \wedge B \to B$

$\vdash (A \to (B \to C)) \to ((A \to B) \to (A \to C))$

$\vdash (A \to C) \to ((B \to C) \to (A \vee B \to C))$

$$\frac{\vdash A \to B}{\vdash \neg A \to \neg B}$$

$\vdash \bot \rightarrow A$

$\vdash (A \rightarrow (B \rightarrow C)) \leftrightarrow (A \wedge B \rightarrow C)$ (Import-Export Law)

$\vdash A \rightarrow (B \rightarrow C) \vdash B \rightarrow (A \rightarrow C)$

$\vdash \neg(A \rightarrow B) \rightarrow \neg B$

$\vdash \neg((A \rightarrow B) \rightarrow B) \rightarrow \neg A$

$$\frac{\vdash A \leftrightarrow B}{\vdash \neg A \leftrightarrow \neg B}$$

- Reasoning on the level of beliefs follows from reasoning on the level of information. The models explicitly represent both levels. Their expressive power is higher then classical approaches.
- Consistent belief sets can be based on inconsistent collections of information.
- Belief sets are always non-trivial if the models are on **consistence-based-situation spaces** (For all $s \in S$ if $s \perp s$ then for some $s' \in S$ it holds that $s' \not\perp s'$ and $s' \sqsubseteq s$).
- The belief sets are not necessarily closed under conjunction.
- Beliefs are not closed under positive or negative introspection.
- The models suggest a non-monotonic, paraconsistent and hyperintensional form of belief and belief change.

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A static belief base model is a five tuple $M = \langle S, V, \circ, \bot, \leq \rangle$ such that,

- S is a non-empty and finite situation space, the situations are denoted by s_i with or without the subscript,
- V is a mapping from S to the power set of set of literals L_l ; $V : S \to P(L_l)$,
- $\circ$ is a binary partial function from SxS to S that satisfies the following conditions:
  - If $s \circ s'$ is defined, it is required that $V_{prop}(s \circ s') \supseteq V_{prop}(s) \cup V_{prop}(s')$.
  - Idempotence: $s \circ s = s$ which means $s \circ s$ is always defined and is equal to s .
  - Commutativity: if $s' \circ s$ is defined, then $s \circ s'$ is also defined and $s' \circ s = s \circ s'$.
  - Partial associativity: if $(s \circ s') \circ s''$ is defined, then $s \circ (s' \circ s'')$ are also defined and $(s \circ s') \circ s'' = s \circ (s' \circ s'')$.
  - Distributivity: Whenever $s \circ s'$ is defined, there is a $s'' \in S$ such that $s \circ s' = s''$ and for all $u \in S$ such that $u \circ s'' = s''$ it holds that $u \circ_D s = s$ or $u \circ_D s' = s'$ or $u = s''$.

- $\perp$ is a binary symmetric relation on S that satisfies the following conditions:
  - If $v \in V(s)$ and $\neg v \in V(s')$ then $s \perp s'$
  - If $s \circ s''$ is defined and $s' \circ s'''$ is defined, if $s \perp s'$ then $s \circ s'' \perp s' \circ s'''$.
- $\leq$ is a total (transitive, reflexive and connected) preorder on $P(S)$,
- For every element s of S, there is a unique $s^* \in S$ *(the star image of s)* with the following features:
  - $V(s^*) = \{\bar{v} | v \notin V(s)$.
  - $s^{**} = s$.
  - $s^* \not\leq s$.
  - If $s' \not\leq s$ then $s' \circ s^*$ is defined, and $s' \circ s^* = s^*$.

A dynamic belief base model is an seven tuple $M = \langle S, V, \circ, \perp, \leq, [\![R\cdot]\!]_G, [\![R\cdot]\!]_L \rangle$ such that,

- $\langle S, V, \circ, \perp, \leq \rangle$ is a static belief base model, and
- $[\![R\cdot]\!]_G$ is a global revision function on M iff $[\![R\cdot]\!]_G$ maps a triple $\langle \phi, M, s \rangle$, where $\phi \in L_D$, M is a dynamic belief base model on S and $s \in S$, to the set X of pairs $\langle M', s' \rangle$ determined uniquely by the following:
  - M' is a dynamic belief base model on S', where $S' = S, V' = V, \circ' = \circ, \perp' = \perp, R' = R, [\![R\cdot]\!]'_M = [\![R\cdot]\!]_M, [\![R\cdot]\!]'_L = [\![R\cdot]\!]_L$, and $s' \in S'$.
  - $\exists u \in S$ such that u is a basic ϕ-situation and $V(M', s') = V(M, s \circ u)$.
  - for all $A, B \subseteq S$, if for some $t \in A$, $t \models \phi$ and for no $t' \in B$, $t' \models \phi$, then $A \leq_{M'} B$; otherwise $A \leq_{M'} B$ iff $A \leq_M B$.

- $[\![R\cdot]\!]_L$ is a local revision function on M iff $[\![R\cdot]\!]_M$ maps a triple $\langle \phi, M, s \rangle$, where $\phi \in L_D$, M is a dynamic belief base model on S and $s \in S$, to the set X of pairs $\langle M', s' \rangle$ determined uniquely by the following:
    - M' is a dynamic belief base model on S', where $S' = S, V' = V, \circ' = \circ, \bot' = \bot, R' = R, [\![R\cdot]\!]'_M = [\![R\cdot]\!]_M, [\![R\cdot]\!]'_L = [\![R\cdot]\!]_L$, and $s' \in S'$.
    - $\exists u \in S$ such that u is a basic ϕ-situation and $V(M', s') = V(M, s \circ u)$
    - for all $A, B \in I \subseteq S$, where $t \in I$ only if $t \sqsubseteq (s \circ u)$, if for some $t \in A$, $t \models \phi$ and for no $t' \in B$, $t' \models \phi$, then $A \leq_{M'} B$; otherwise $A \leq_{M'} B$ iff $A \leq_M B$.