Perfect Sets and Games on Generalized Baire Spaces

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Generalized Baire spaces

Let κ be an uncountable cardinal such that $\kappa^{<\kappa} = \kappa$.

The κ-Baire space ${}^{\kappa}\kappa$ is the set of functions $f : \kappa \to \kappa$, with the bounded topology: basic open sets are of the form

$$N_s = \{f \in {}^{\kappa}\kappa : s \subset f\}, \qquad \text{where } s \in {}^{<\kappa}\kappa.$$

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The κ-Cantor space ${}^{\kappa}2$ is defined similarly.

κ-Borel sets: close the family of open subsets under intersections and unions of size $\leq \kappa$ and complementation.

Perfect and scattered subsets of the κ-Baire space

Definition (Väänänen, 1991)

Let $X \subseteq {}^\kappa\kappa$, let $x_0 \in {}^\kappa\kappa$ and let $\omega \leq \gamma \leq \kappa$. The game $\mathcal{V}_\gamma(X, x_0)$ has length γ and is played as follows:

I		U_1		$\dots$		U_α		$\dots$
II	x_0		x_1		$\dots$		x_α		$\dots$

II first plays x_0. In each round $0 < \alpha < \gamma$, **I** plays a basic open subset U_α of X, and then **II** chooses

$$x_\alpha \in U_\alpha \text{ with } x_\alpha \neq x_\beta \text{ for all } \beta < \alpha.$$

I has to play so that $U_{\beta+1} \ni x_\beta$ in each successor round $\beta + 1 < \gamma$ and $U_\alpha = \bigcap_{\beta<\alpha} U_\beta$ in each limit round $\alpha < \gamma$.

II wins a given run of the game if she can play legally in all rounds $\alpha < \gamma$.

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X is a γ-perfect set if X is closed and $\mathbf{II}$ wins $\mathcal{V}_\gamma(X, x_0)$ for all $x_0 \in X$.

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- X is ω-perfect iff X is perfect in the usual sense (i.e., iff X closed and has no isolated points).
- X is ω-scattered iff X is scattered in the usual sense (i.e., each nonempty subspace contains an isolated point).
- $\mathcal{V}_\gamma(X, x_0)$ may not be determined when $\gamma > \omega$.

A different definition of κ-perfectness

A subset of ${}^{\kappa}\kappa$ is closed iff it is the set of branches

$$[T] = \{x \in {}^{\kappa}\kappa : x \restriction \alpha \in T \text{ for all } \alpha < \kappa\}$$

of a subtree T of ${}^{<\kappa}\kappa$.

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Definition

A subtree T of ${}^{<\kappa}\kappa$ is a strongly κ-perfect tree if T is $<\kappa$-closed and every node of T extends to a splitting node.

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Example (Huuskonen)

The following set is κ-perfect but is not strongly κ-perfect:

$$Y_\omega = \{x \in {}^{\kappa}3 : |\{\alpha < \kappa : x(\alpha) = 2\}| < \omega\}.$$

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Proposition

Let X be a closed subset of ${}^{\kappa}\kappa$.

$$X \textit{ is } \kappa\textit{-perfect} \iff X = \bigcup_{i \in I} X_i \textit{ for strongly } \kappa\textit{-perfect sets } X_i.$$

Perfect and scattered trees

Definition (Galgon, 2016)

Let T be a subtree of ${}^{<\kappa}2$, let $t \in T$, and let $\omega \leq \gamma \leq \kappa$. The game $\mathcal{G}_\gamma(T,t)$ has length γ and is played as follows:

I	δ_0		i_0	$\dots$	δ_α		i_α	$\dots$
II		t_0		$\dots$		t_α		$\dots$

In each round $\alpha < \gamma$, player **I** first plays $\delta_\alpha < \kappa$. Then **II** plays a node $t_\alpha \in T$ of height $\geq \delta_\alpha$, and **I** chooses $i_\alpha < 2$. **II** has to play so that $t \subseteq t_0$, and

$$t_\beta {}^\frown \langle i_\beta \rangle \subseteq t_\alpha \text{ for all } \beta < \alpha < \gamma.$$

II wins a given run of the game if she can play legally in all rounds $\alpha < \gamma$.

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Let T be a subtree of ${}^{<\kappa}\kappa$.

1. T *is a* κ*-perfect tree* $\iff$ $[T]$ *is a* κ*-perfect set.*

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Let T be a subtree of ${}^{<\kappa}\kappa$.

1. $$T \text{ is a } \kappa\text{-perfect tree} \iff [T] \text{ is a } \kappa\text{-perfect set.}$$
2. *If the κ-perfect set property holds for closed subsets of ${}^{\kappa}\kappa$ (i.e., every closed subset of size $> \kappa$ has a κ-perfect subset), then*

   $$T \text{ is a } \kappa\text{-scattered tree} \iff [T] \text{ is a } \kappa\text{-scattered set.}$$

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$$T \text{ is a } \kappa\text{-scattered tree} \iff [T] \text{ is a } \kappa\text{-scattered set.}$$

Remark: The κ-PSP for closed subsets of ${}^{\kappa}\kappa$ is equiconsistent with the existence of an inaccessible cardinal $\lambda > \kappa$.

γ-perfect sets and trees when $\gamma \leq \kappa$

Theorem (Sz)

Let T be a subtree of ${}^{<\kappa}\kappa$ and let $\omega \leq \gamma \leq \kappa$.

1. *If $[T]$ is a γ-perfect set, then T is a γ-perfect tree.*

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3. *If κ is weakly compact and $T \subseteq {}^{<\kappa}2$, then*

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More generally: this holds if κ has the tree property and T is a κ-tree.

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More generally: this holds if κ has the tree property and T is a κ-tree.

Question

Is it consistent that 3 holds for "scattered" instead of "perfect"?

Theorem (Sz)

An analogue of the previous theorem holds for the levels of the "generalized Cantor-Bendixson hierarchies" associated to subsets of ${}^{\kappa}\kappa$ *and to subtrees of* ${}^{<\kappa}\kappa$.

- See the next 3 slides for definitions and a precise statement this theorem.

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An analogue of the previous theorem holds for the levels of the "generalized Cantor-Bendixson hierarchies" associated to subsets of ${}^{\kappa}\kappa$ and to subtrees of ${}^{<\kappa}\kappa$.

- See the next 3 slides for definitions and a precise statement this theorem.
- Generalized Cantor-Bendixson hierarchies can be defined for subsets of ${}^{\kappa}\kappa$ and for subtrees of ${}^{<\kappa}\kappa$, using modifications of Väänänen's and Galgon's games.
- For subtrees of ${}^{<\kappa}\kappa$, modifications of a game equivalent to $\mathcal{G}_\gamma(T,t)$ need to be used.

Definition (Hyttinen; Väänänen)

Let $X \subseteq {}^\kappa\kappa$, let $x_0 \in {}^\kappa\kappa$, and let S be a tree without branches of length $\geq \kappa$. The S-approximation $\mathcal{V}_S(X, x_0)$ of $\mathcal{V}_\kappa(X, x_0)$ is the following game.

I		s_1, U_1		$\dots$		s_α, U_α		$\dots$	
II	x_0		x_1		$\dots$		x_α		$\dots$

In each round $\alpha > 0$, **I** first plays $s_\alpha \in S$ such that $s_\alpha >_S s_\beta$ for all $0 < \beta < \alpha$. Then **I** plays U_α and **II** plays x_α according to the same rules as in $\mathcal{V}_\kappa(X, x_0)$. The first player who can not move loses, and the other player wins.

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Generalizing the Cantor-Bendixson hierarchy

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$$\mathrm{Sc}_S(X) = \{x \in X : \textbf{I} \text{ wins } \mathcal{V}_S(X, x)\};$$

$$\mathrm{Ker}_S(X) = \{x \in {}^\kappa\kappa : \textbf{II} \text{ wins } \mathcal{V}_S(X, x)\}.$$

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The sets $X \cap \mathrm{Ker}_S(X)$ (resp. $X - \mathrm{Sc}_S(X)$) can be seen as the "levels of a generalized Cantor-Bendixson hierarchy" for the set X associated to **II** (resp. **I**).[1]

[1] For a precise verison of this statement, see: J. Väänänen. A Cantor-Bendixson theorem for the space ${\omega_1}^{\omega_1}$. *Fund. Math.* 137:187–199, 1991.

Generalizing the Cantor-Bendixson hierarchy

Proposition (Sz.)

There exists a family

$$\{\mathcal{G}'_\kappa(T,t) : T \text{ is a subtree of } {}^{<\kappa}\kappa,\ t \in T \text{ and } \omega \leq \gamma \leq \kappa\}$$

of games such that the following hold for all T, t and γ.

- *The games $\mathcal{G}'_\gamma(T,t)$ and $\mathcal{G}_\gamma(T,t)$ are equivalent whenever $T \subseteq {}^{<\kappa}2$.*
- *Given a tree S without branches of length $\geq \kappa$, let $\mathcal{G}'_S(T,t)$ denote the S-approximation of $\mathcal{G}'_\kappa(T,t)$ (this is defined analogously to the S-approximations $\mathcal{V}_S(T,x)$). Let*

  $$\mathrm{Sc}_S(T) = \{t \in T : \mathbf{I} \text{ wins } \mathcal{G}'_S(T,t)\};$$
  $$\mathrm{Ker}_S(T) = \{t \in T : \mathbf{II} \text{ wins } \mathcal{G}'_S(T,t)\}.$$

  Then $\mathrm{Ker}_S(T)$ (resp. $T - \mathrm{Sc}_S(T)$) generalize the levels of the Cantor-Bendixson hierarchy for T which was defined by Galgon, for $\mathbf{II}$ (resp. $\mathbf{I}$).[2]

[2] In the same sense that $X \cap \mathrm{Ker}_S(X)$ (resp. $X - \mathrm{Sc}_S(X)$) generalize the levels of the Cantor-Bendixson hierarchy for X, for $\mathbf{II}$ (resp. $\mathbf{I}$).

Theorem (Sz; precise version of previous theorem)

Let T be a subtree of ${}^{<\kappa}\kappa$; let S be a tree without branches of length $\geq \kappa$.

1. $\mathrm{Ker}_S([T]) \subseteq [\mathrm{Ker}_S(T)]$.
2. $[T] - \mathrm{Sc}_S([T]) \subseteq [T - \mathrm{Sc}_S(T)]$.
3. *If κ has the tree property and T is a κ-tree, then*

$$\mathrm{Ker}_S([T]) = [\mathrm{Ker}_S(T)].$$

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Theorem (Väänänen, 1991)

The following Cantor-Bendixson theorem for ${}^{\kappa}\kappa$ is consistent relative to the existence of a measurable cardinal $\lambda > \kappa$:

Every closed subset of ${}^{\kappa}\kappa$ is the (disjoint) union of

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Remark: The κ-PSP for closed subsets of ${}^{\kappa}\kappa$ is equiconsistent with the existence of an inaccessible cardinal $\lambda > \kappa$.

Proof (idea).

Let X be a closed subset of ${}^{\kappa}\kappa$. Its set of κ-condensation points is defined to be

$$CP_{\kappa}(X) = \{x \in X : |X \cap N_{x \restriction \alpha}| > \kappa \text{ for all } \alpha < \kappa\}.$$

If the κ-PSP holds for closed subsets of ${}^{\kappa}\kappa$, then $CP_{\kappa}(X)$ is a κ-perfect set and $X - CP_{\kappa}(X)$ is a κ-scattered set of size $\leq \kappa$. □

Thank you!