Antichains of copies of ultrahomogeneous structures

Boriša Kuzeljević

University of Novi Sad

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Joint work with Miloš Kurilić

Relational structures

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Embedding from $\mathbb{X} = \langle X, \{\rho_i : i \in I\}\rangle$ into $\mathbb{Y} = \langle Y, \{\sigma_i : i \in I\}\rangle$:

1-1 mapping $f : X \to Y$ such that

$$\bar{x} \in \rho_i \Leftrightarrow f(\bar{x}) \in \sigma_i$$

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$\mathrm{Emb}(\mathbb{X}, \mathbb{Y})$ denotes the set of all embeddings from $\mathbb{X}$ into $\mathbb{Y}$.
$\mathrm{Aut}(\mathbb{X})$ denotes the set of all automorphisms of a structure $\mathbb{X}$.

Relational structures

Stabilizer

Suppose that $\mathbb{X}$ is a relational structure, and that F is a finite subset of $\mathbb{X}$. Stabilizer (pointwise) of the set F in $\mathrm{Aut}(\mathbb{X})$ is the group

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Orbits

Suppose that $\mathbb{X}$ is a relational structure, that F is a finite subset of $\mathbb{X}$, and that $x \in \mathbb{X}$. Orbit of the point x with respect to $\mathrm{Aut}_F(\mathbb{X})$ is the set:

$$\mathrm{orb}_{\mathrm{Aut}_F(\mathbb{X})}(x) = \mathrm{orb}_F(x) = \{y \in \mathbb{X} \setminus F : (\exists g \in \mathrm{Aut}_F(\mathbb{X}))\ g(x) = y\}.$$

Ultrahomogeneous structures

Definition:

A relational structure $\mathbb{X}$ is *ultrahomogeneous* if for every isomorphism $\varphi : \mathbb{A} \to \mathbb{B}$ between finite substructures $\mathbb{A}$ and $\mathbb{B}$ of $\mathbb{X}$, there is an automorphism $\psi \in \mathrm{Aut}(\mathbb{X})$ such that $\psi \restriction \mathbb{A} = \varphi$.

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Lemma

Suppose that $\mathbb{X}$ is a countable ultrahomogeneous relational structure, and that $\mathbb{A}$ is a substructure of $\mathbb{X}$. Then $\mathbb{A} \cong \mathbb{X}$ if and only if

$$(\forall F \in [\mathbb{A}]^{<\omega})\ (\forall x \in \mathbb{X} \setminus F)\ \mathrm{orb}_F(x) \cap A \neq \emptyset.$$

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Strong amalgamation property

A countable ultrahomogeneous relational structure $\mathbb{X}$ satisfies *the strong amalgamation property* (SAP) if $\mathbb{X} \setminus F \cong \mathbb{X}$ for each finite set $F \subset \mathbb{X}$.

If $\mathbb{X}$ is a relational structure, then $\mathbb{P}(\mathbb{X})$ denotes the set:

$$\mathbb{P}(\mathbb{X}) = \{A \subset X : \mathbb{A} \cong \mathbb{X}\} = \{f[X] : f \in \mathrm{Emb}(\mathbb{X}, \mathbb{X})\}.$$

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Theorem

Suppose that $\mathbb{X}$ is an ultrahomogeneous structure with SAP. Then there is a partition of $\mathbb{X}$ into countably many copies whose intersection with each orbit of $\mathbb{X}$ is infinite.

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Definition

Suppose that P is a poset. We say that $A \subset P$ is an antichain if for arbitrary distinct $x, y \in A$, there is no $z \in P$ such that $z \leq x$ and $z \leq y$.

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Problem:

Is there a poset P such that $\mathfrak{a}(P \times P) < \mathfrak{a}(P)$?
$\mathfrak{a}(P) = \min\{|\mathcal{A}| : \mathcal{A}$ is an infinite maximal antichain in $P\}$.

Maximal antichains I

Let $\mathbb{K}_\omega$ be a countable complete graph.

MAD families

- There is a maximal antichain of size $\mathfrak{c}$ in the poset $\langle \mathbb{P}(\mathbb{K}_\omega), \subset \rangle$.
- There is no countable maximal antichain in the poset $\langle \mathbb{P}(\mathbb{K}_\omega), \subset \rangle$.

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Theorem (Kurilić - Marković 2015)

There is a countable maximal antichain in $\langle \mathbb{P}(\mathbb{G}_{\mathsf{Rado}}), \subset \rangle$.

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Theorem (Kurilić 2014)

Let $\mathbb{X}$ be a countable indivisible relational structure. Then the poset $\mathbb{P}(\mathbb{X})$ contains an almost disjoint family of size $\mathfrak{c}$.

Theorem

Suppose that $\mathbb{X}$ is a countable ultrahomogeneous structure in a finite relational language. If $\mathbb{X}$ satisfies SAP, then $\mathbb{P}(\mathbb{X})$ contains an almost disjoint family of size $\mathfrak{c}$, consisting of copies whose intersection with each orbit of $\mathbb{X}$ is infinite.
In particular, there is a maximal antichain of size $\mathfrak{c}$ in $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$.

Countable ultrahomogeneous posets

Schmerl 1979

A countable partial order is ultrahomogeneous if and only if it is isomorphic to one of the orders from the following list:

- random poset $\mathbb{D}$;
- $\mathbb{B}_n$ for $1 \leq n \leq \omega$;
- $\mathbb{C}_n$ for $1 \leq n \leq \omega$;
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There is a countable maximal antichain in $\langle \mathbb{Q}, \subset \rangle$.

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Lemma

There is a countable maximal antichain in $\langle \mathbb{Q}, \subset \rangle$.

Theorem

Suppose that $1 \leq n \leq \omega$. There are maximal antichains both of size $\mathfrak{c}$ and ω in $\langle \mathbb{P}(\mathbb{C}_n), \subset \rangle$.

Products of posets

Lemma

Let $n < \omega$, and let $\{P_m : m < n\}$ be a collection of posets. Suppose that P_m has a maximum 1_m for each $m < n$, and that there is some $k < n$ such that P_k contains a countable maximal antichain. Then there is a countable maximal antichain in $\prod_{m<n} P_m$.

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Corollary:

Let $n < \omega$. There are maximal antichain of size both ω and $\mathfrak{c}$ in the poset $\langle \mathbb{P}(\mathbb{B}_n), \subset \rangle$.

Products of posets

Definition

Suppose that $\{P_i : i \in I\}$ is a collection of posets with maximum 1_i, and minimum 0_i $(i \in I)$. Countable support product of P_i's is:

$$\prod_{i\in I}^{CS} P_i = \left\{x \in \prod_{i\in I} P_i : |\mathrm{supp}(x)| \geq \omega\right\}.$$

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Corollary:

There is no countable maximal antichain in $\langle \mathbb{P}(\mathbb{B}_\omega), \subset\rangle$.

Random poset

Theorem

There are maximal antichains of size both ω and $\mathfrak{c}$ in $\langle \mathbb{P}(\mathbb{D}), \subset \rangle$.