A Complete Intuitionistic Temporal Logic for Topological Dynamics

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Joint work with J. Boudou and M. Diéguez

Why intuitionistic temporal logic?

Intuitionistic temporal logics have been suggested for

1. **Davies 1996:** Typing partially evaluated programs
2. **Maier 2004:** Possibly terminating reactive systems
3. **Kremer 2004:** Reasoning about topological dynamics
4. **Cabalar and Pérez 2007:** Temporal answer-set programming (based on Here-and-There logic)

Today we will focus on 3.

Modal logic for topological dynamics

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Kremer, Mints 2005: Observed that adding $\Box$ to the language allowed us to reason about asymptotic behavior

Case study: The Poincaré recurrence theorem

Theorem (Poincaré)

Let $X \subseteq \mathbb{R}^n$ be open and bounded and let $S\colon X \to X$ be volume-preserving; that is,

$$\mathrm{vol} \circ S^{-1} \equiv \mathrm{vol}$$

Then, if $E \subseteq X$ is open and non-empty, it follows that E has a recurrent point; that is, there is $x \in E$ and $n > 0$ such that $S^n(x) \in E$

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Theorem (Kremer, Mints)

Poincaré recurrence is equivalent to the validity of

$$\blacksquare p \to \blacklozenge \circ \diamond p$$

Good news and bad news

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DFD 2012: The logic over $\mathcal{L}_{\blacksquare\circ\square}$ admits a natural axiomatization when enriched with the tangled closure modality

Kremer's intuitionistic temporal logic

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However, the following standard validities fail

- $\Box p \rightarrow \circ\Box p$
- $\Box\circ p \rightarrow \circ\Box p$
- $\Box p \rightarrow \Box\Box p$

Topological semantics for intuitionistic logic

Models

$\mathcal{M} = (X, \mathcal{T}, V)$, where:

- $(X, \mathcal{T})$ is a topological space
- $V \colon \mathbb{PV} \to \mathcal{T}$

Topological semantics for intuitionistic logic

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$\mathcal{M} = (X, \mathcal{T}, V)$, where:

- $(X, \mathcal{T})$ is a topological space
- $V : \mathbb{PV} \to \mathcal{T}$

Truth sets

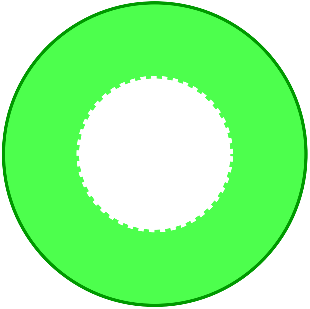
- $[\![\bot]\!] = \varnothing$
- $[\![p]\!] = V(p)$
- $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$
- $[\![\varphi \vee \psi]\!] = [\![\varphi]\!] \cup [\![\psi]\!]$
- $[\![\varphi \to \psi]\!]$

  $= \left([\![\varphi]\!]^c \cup [\![\psi]\!]\right)^\circ$

Interior of $A \subseteq X$:

$$A^\circ = \bigcup \{U \in \mathcal{T} : U \subseteq A\}$$

Classical regions



$[\![P]\!]$

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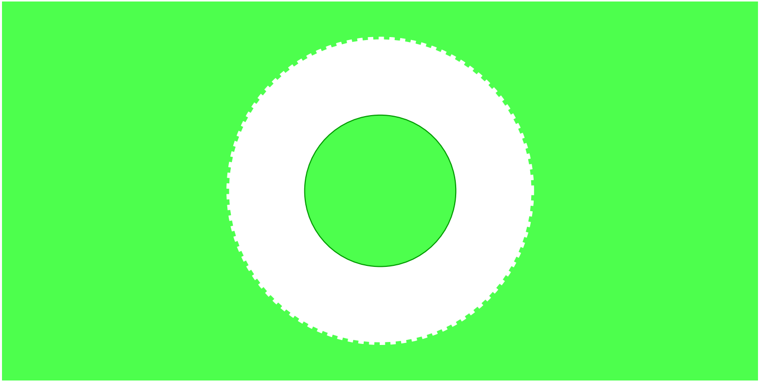

$[\![\neg P]\!]$

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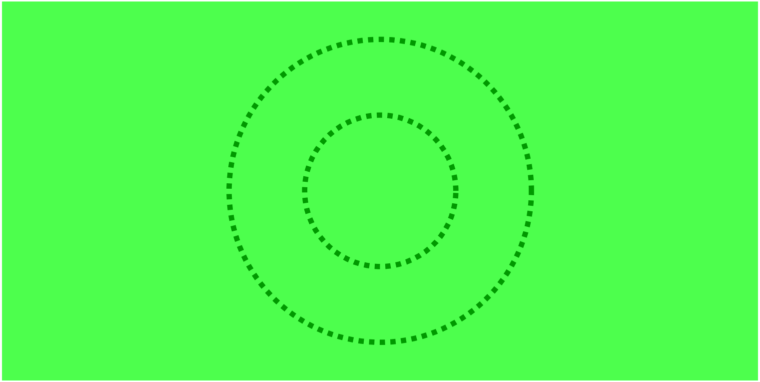
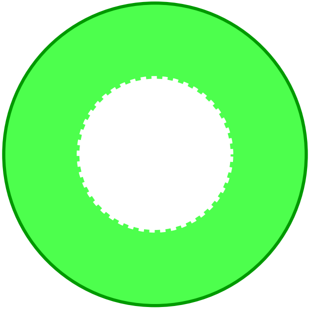


$$[\![P \vee \neg P]\!]$$

Intuitionistic regions



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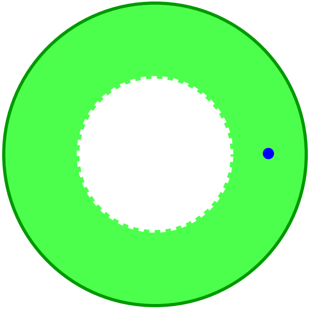

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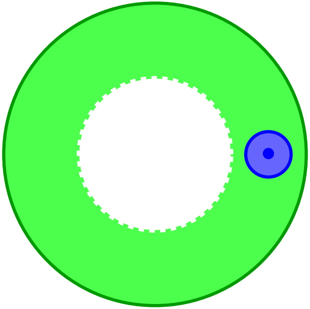

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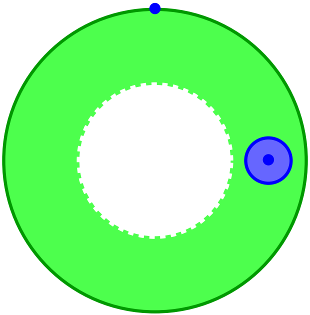

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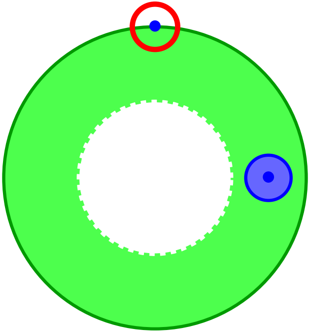

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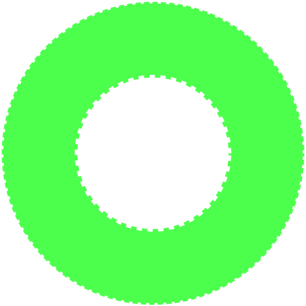

$[\![P]\!]^I$

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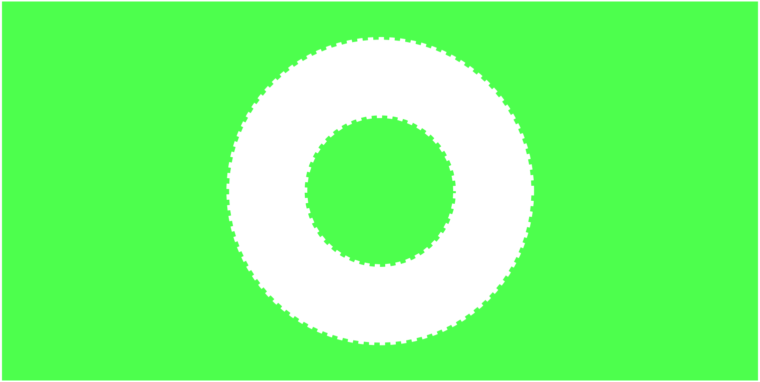

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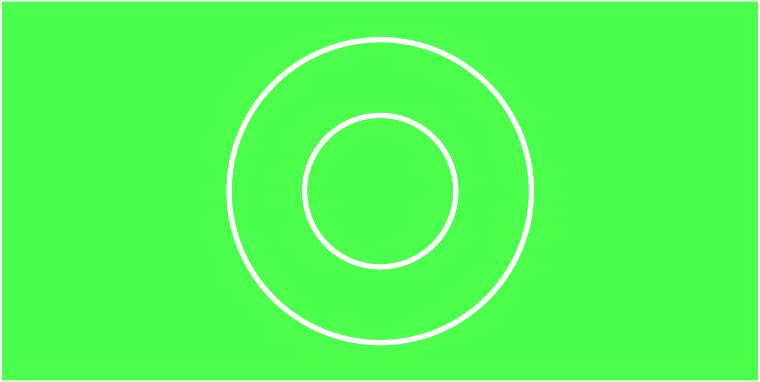

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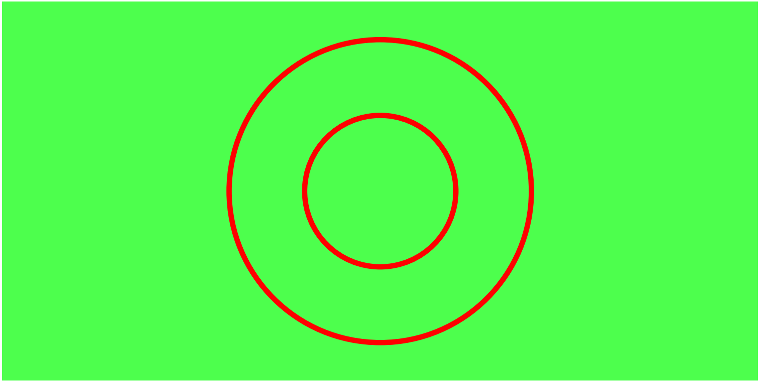

$[\![P \vee \neg P]\!]^I$ Fails!

Special case: Poset models

Definition
A partial order $\preccurlyeq$ on a set W generates the topology $\mathcal{T}_{\preccurlyeq}$ on W where $U \subseteq W$ is open if $w \in U$ and $v \succcurlyeq w$ implies $v \in U$

Lemma
If $(X, \mathcal{T}, V)$ *is a model such that* T *is generated by a partial order* $\preccurlyeq$, *then*

$$(\mathcal{M}, w) \models \varphi \to \psi \textit{ iff } \forall v \succcurlyeq w\, (\mathcal{M}, v) \not\models \varphi \textit{ or } (\mathcal{M}, v) \models \psi$$

Intuitionistic temporal logic

Language $\mathcal{L}_{\diamond\Box\forall}$: $\varphi, \psi :=$

$$p \mid \bot \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \to \psi \mid \circ\varphi \mid \diamond\varphi \mid \Box\varphi \mid \forall\varphi$$

Models: $(X, \mathcal{T}, S, V)$, where $S\colon X \to X$ is continuous

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Truth of temporal operators

$$[\![\circ\varphi]\!] = S^{-1}[\![\varphi]\!]$$

$$[\![\diamond\varphi]\!] = \textstyle\bigcup_{n<\omega} S^{-n}[\![\varphi]\!]$$

$$[\![\Box\varphi]\!] = \left(\textstyle\bigcap_{n<\omega} S^{-n}[\![\varphi]\!]\right)^{\circ}$$

$$[\![\forall\varphi]\!] = \begin{cases} X & \text{if } [\![\varphi]\!] = X \\ \varnothing & \text{otherwise} \end{cases}$$

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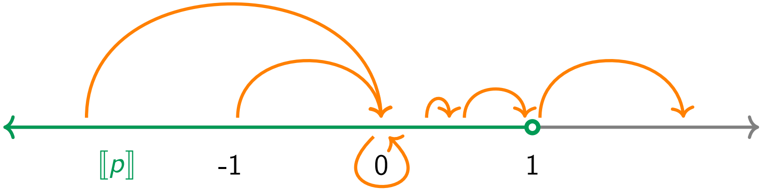
Dynamical posets: If $\mathcal{T}$ is generated by $\preccurlyeq$, S is continuous iff $w \preccurlyeq v$ implies $S(w) \preccurlyeq S(v)$

Kremer's counterexample: $\Box p \to \circ\Box p$ fails!

- $X = \mathbb{R}$
- $V(p) = (-\infty, 1)$
- $S(x) = \begin{cases} 2x & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$



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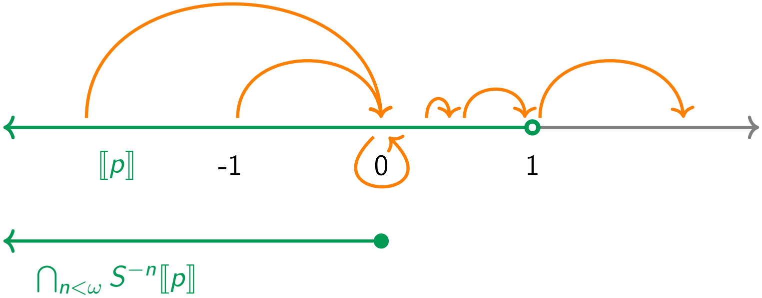

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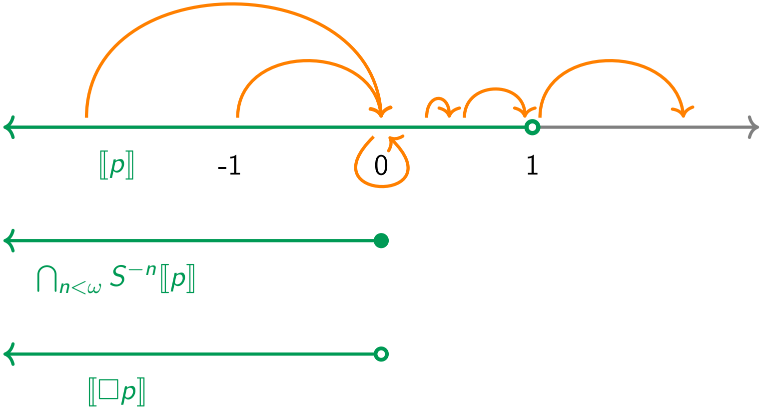

Some good news

Theorem (DFD)

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Theorem (Boudou, Diéguez, DFD)

The validity problem for $\mathcal{L}_{\Diamond\Box}$ *is decidable over the class of dynamical posets*

The calculus $\mathsf{ITL}^0_\Diamond$

ITaut	Standard intuitionistic propositional axioms

Temporal axioms:

$\mathsf{Next}_\bot$	$\neg \circ \bot$
$\mathsf{Next}_\wedge$	$(\circ\varphi \wedge \circ\psi) \to \circ(\varphi \wedge \psi)$
$\mathsf{Next}_\vee$	$\circ(\varphi \vee \psi) \to (\circ\varphi \vee \circ\psi)$
$\mathsf{Next}_\to$	$\circ(\varphi \to \psi) \to (\circ\varphi \to \circ\psi)$
$\mathsf{Fix}_\Diamond$	$(\varphi \vee \circ \Diamond \varphi) \to \Diamond\varphi$

Rules:

MP $\quad \dfrac{\varphi \quad \varphi \to \psi}{\psi}$ $\qquad$ $\mathsf{Nec}_\circ \quad \dfrac{\varphi}{\circ\varphi}$

$\mathsf{Mon}_\Diamond \quad \dfrac{\varphi \to \psi}{\Diamond\varphi \to \Diamond\psi}$ $\qquad$ Ind $\quad \dfrac{\circ\varphi \to \varphi}{\Diamond\varphi \to \varphi}$

The calculus $\mathsf{ITL}^0_{\diamond\forall}$

Add the following to $\mathsf{ITL}^0_{\diamond}$:

$\mathsf{K}_\forall$	$\forall(\varphi \to \psi) \to (\forall\varphi \to \forall\psi)$	$\mathsf{EM}_\forall$	$\forall\varphi \vee \neg\forall\varphi$
$\mathsf{Dist}_\forall$	$\forall(\varphi \vee \forall\psi) \to \forall\varphi \vee \forall\psi$	$\mathsf{T}_\forall$	$\forall\varphi \to \varphi$
$\mathsf{Next}_\forall$	$\forall\varphi \leftrightarrow \circ\forall\varphi$	$4_\forall$	$\forall\varphi \to \forall\forall\varphi$
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- for the class of dynamical posets?

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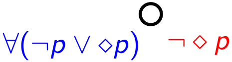

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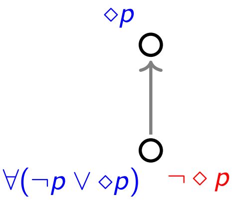

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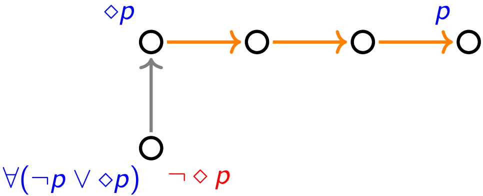

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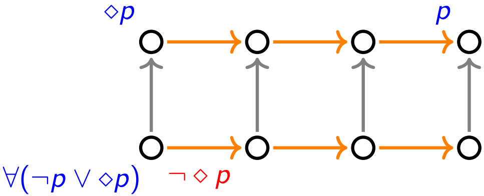

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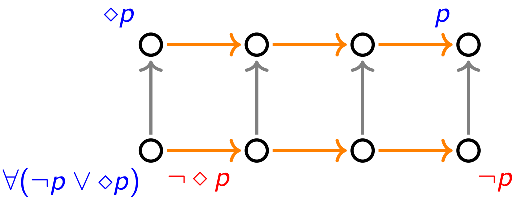

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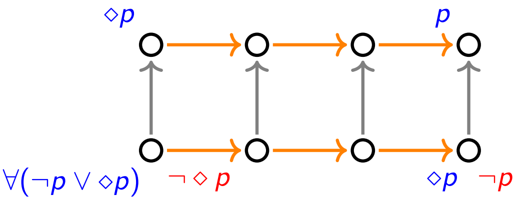

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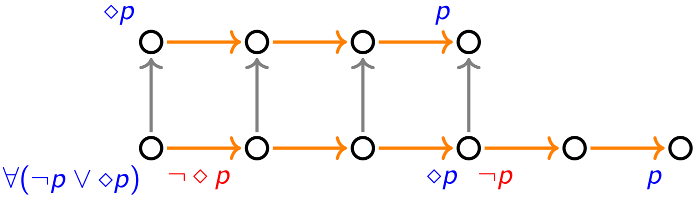

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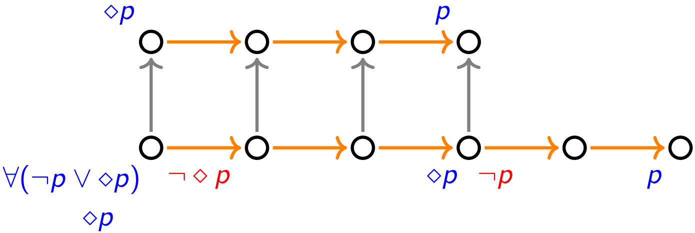
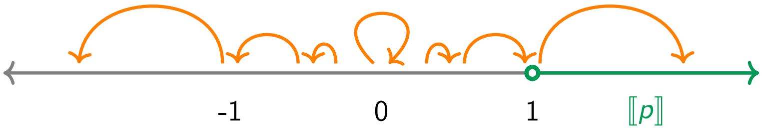


Falsifying $\forall(\neg p \vee \diamond p) \rightarrow (\diamond p \vee \neg \diamond p)$ topologically

- $X = \mathbb{R}$
- $V(p) = (1, \infty)$
- $S(x) = 2x$



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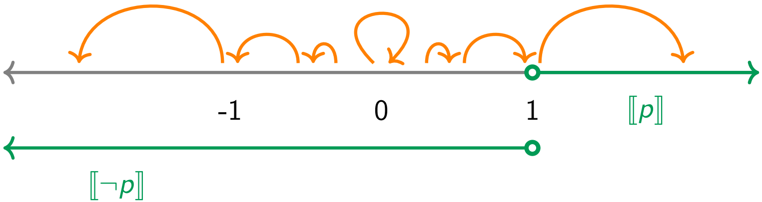

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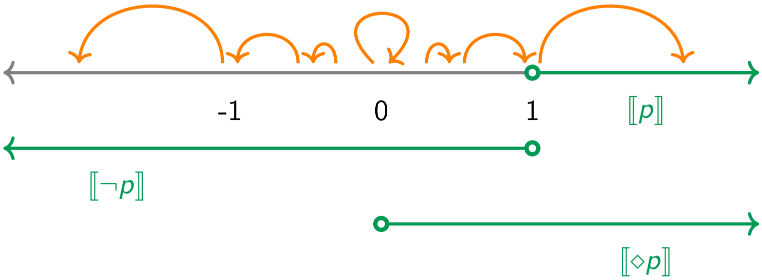

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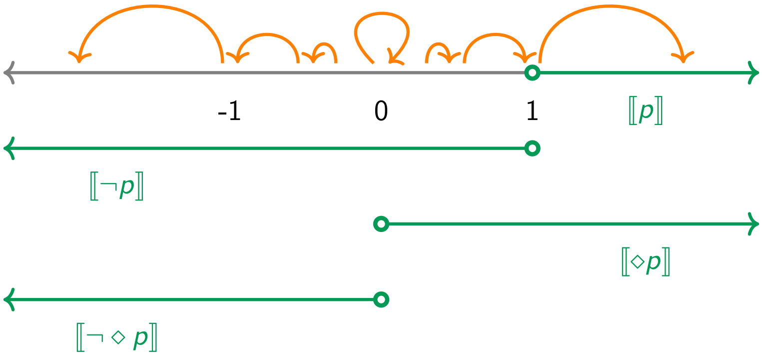

Conservativity for ‘eventually’

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Proof idea:

Formulas not containing $\forall$ are made true in a finite amount of time and hence we may ‘discretize’ models.

Completeness

Theorem (Boudou, Diéguez, DFD)

If $\varphi \in \mathcal{L}_{\diamond\forall}$ *is valid on the class of dynamical systems then* $\mathsf{ITL}^0_{\diamond\forall} \vdash \varphi$.

Completeness

Theorem (Boudou, Diéguez, DFD)

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Theorem (Boudou, Diéguez, DFD)

If $\varphi \in \mathcal{L}_{\diamond}$ is valid on the class of dynamical **posets** *then $\mathsf{ITL}^0_{\diamond} \vdash \varphi$.*

Gödel-Tarski translation

The translation $\varphi \mapsto \varphi^{\blacksquare}$ embeds $\mathcal{L}_{\Diamond\Box}$ into the classical $\mathcal{L}_{\blacksquare\Box}$ by setting

- $p^{\blacksquare} = \blacksquare p$
- $(\varphi \wedge \psi)^{\blacksquare} = \varphi^{\blacksquare} \wedge \psi^{\blacksquare}$
- $(\varphi \to \psi)^{\blacksquare} = \blacksquare(\varphi^{\blacksquare} \to \psi^{\blacksquare})$
- $(\Diamond\varphi)^{\blacksquare} = \Diamond\varphi^{\blacksquare}$
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- $\bot^{\blacksquare} = \bot$
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Theorem

Given $\varphi \in \mathcal{L}_{\Diamond\Box\forall}$, φ is valid iff $\varphi^{\blacksquare}$ is.

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Theorem

Given $\varphi \in \mathcal{L}_{\diamond\Box\forall}$, φ is valid iff $\varphi^{\blacksquare}$ is.

Corollary

The set of $\mathcal{L}_{\diamond\Box}$-formulas valid over the class of dynamical systems is computably enumerable.

Dynamic topological logic on metric spaces

Theorem (DFD)

Given $n \geq 2$, every formula of $\mathcal{L}_{\blacksquare\Box}$ that is (classically) satisfiable on a dynamical poset is satisfiable on $\mathbb{R}^n$.

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Intuitionistic temporal logic on metric spaces

Theorem (Boudou, Diéguez, DFD)

If $\varphi \in \mathcal{L}_\diamond$ *is valid on*

- $\mathbb{R}^n$ *for any fixed* $n \geq 2$*, or*
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Theorem (Boudou, Diéguez, DFD)

If $\varphi \in \mathcal{L}_{\diamond\forall}$ is valid on $\mathbb{Q}$ then $\mathsf{ITL}^0_{\diamond\forall} \vdash \varphi$.

Future work

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Děkuji!