Towards a classification of algebraizable FDE-based modal logics

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Logic Colloquium,
16 August 2019

Preliminaries

Some systems

NK^{-} is a modal logic over Nelson's N4

i) S. Odintsov, H. Wansing, 2004;

ii) the language is $\wedge, \vee, \sim, \rightarrow_i, \Box, \Diamond$;

iii) a number of extensions with different "semantic dualities".

KN4 is a modal logic with strong *classical* implication $\Rightarrow_c$

i) L. Goble, 2006;

ii) the language is $\wedge, \vee, \sim, \Rightarrow_c, \Box$;

iii) Hilbert-style axiom system with infinite set of rules schemata.

Some systems

$\mathsf{K_{fde}}$ is an FDE-based version of K

i) G. Priest, 2008;
ii) the language is $\wedge, \vee, \sim, \Box$ (no conditionals!);
iii) intended to be the minimal extension of FDE with $\Box$;

MBL is the modal *bilattice* logic

i) A. Jung, U. Rivieccio, 2013;
ii) the language is $\wedge, \vee, \otimes, \oplus, T, F, N, B, \rightarrow_c, \Box$;
iii) a modal extension of billatice logic;
iv) a very unorthodox modality.

Some systems

BK is a Belnapian version of K

i) S. Odintsov, H. Wansing, 2010;
ii) the language is $\wedge, \vee, \sim, F, \Box, \Diamond$;
iii) an extension of K_{fde} with *classical* implication $\rightarrow_c$, F and $\Diamond$.

BK^{FS} is a Fischer Servi style Belnapian modal logic

i) S. Odintsov, H. Wansing, 2017;
ii) the language is $\wedge, \vee, \sim, \rightarrow_c, F, \Box, \Diamond$;
iii) motivated by the standard translation into first-order language.

To summarize

These systems

- arise from different motivations;
- have different non-modal languages;
- different styles of axiomatics;
- different styles of semantics;
- yet...
- have veeeery similar modalities;
- extend first-degree entailment FDE.

First-degree entailment

FDE

A. Anderson, N. Belnap (1962) *Tautological entailments*

Axioms

*a*1. $\varphi \wedge \psi \vdash \varphi$;

*a*2. $\varphi \wedge \psi \vdash \psi$;

*a*3. $\varphi \vdash \varphi \vee \psi$;

*a*4. $\psi \vdash \varphi \vee \psi$;

*a*5. $\varphi \wedge (\psi \vee \chi) \vdash (\varphi \wedge \psi) \vee \chi$;

*a*6. $\varphi \dashv\vdash {\sim}{\sim}\varphi$;

*a*7. ${\sim}(\varphi \wedge \psi) \dashv\vdash {\sim}\varphi \vee {\sim}\psi$;

*a*8. ${\sim}(\varphi \vee \psi) \dashv\vdash {\sim}\varphi \wedge {\sim}\psi$;

Rules

$$\frac{\varphi \vdash \psi \qquad \psi \vdash \chi}{\varphi \vdash \psi}\;; \quad \frac{\varphi \vdash \chi \qquad \psi \vdash \chi}{(\varphi \vee \psi) \vdash \chi}\;; \quad \frac{\chi \vdash \varphi \qquad \chi \vdash \psi}{\chi \vdash (\varphi \wedge \psi)}\,.$$

Belnapian interpretation

Consider classical truth values t and f.

$$\text{From} \quad v(p) \in \{t, f\} \quad \text{to} \quad V(p) \subseteq \{t, f\}.$$

Compute classically all combinations:

$$
\begin{aligned}
t \in V(\varphi \wedge \psi) &\iff t \in V(\varphi) \text{ and } t \in V(\psi);\\
f \in V(\varphi \wedge \psi) &\iff t \in V(\varphi) \text{ or } t \in V(\psi);\\
t \in V(\varphi \vee \psi) &\iff f \in V(\varphi) \text{ or } f \in V(\psi);\\
f \in V(\varphi \vee \psi) &\iff f \in V(\varphi) \text{ and } f \in V(\psi);\\
t \in V({\sim}\varphi) &\iff f \in V(\varphi);\\
f \in V({\sim}\varphi) &\iff t \in V(\varphi).
\end{aligned}
$$

Put

$$\varphi \vdash_{\mathsf{FDE}} \psi \iff \forall V\, (t \in V(\varphi) \Longrightarrow t \in V(\psi)).$$

Some remarks

We consider FDE as a system in the langauge $\{\wedge, \vee, \sim\}$ as opposed to a system with implication but no nesting.

This way FDE does not have theorems, hence formula-formula sequents.

Gives rise to a four-valued modal framework.

FDE has an alternative characterization with contraposition as a rule of inference:

$$\frac{\varphi \vdash \psi}{\sim\psi \vdash \sim\varphi}$$

A classification of FDE-based modal logics

Four-valued framework

A. Bochman (1998) *Biconsequence relations*

> *"Due to the correspondence between four-valued interpretations and their bicomponent representation, any four-valued connective $\sharp(A_1, \ldots, A_n)$ can always be determined by a pair of conditions describing, respectively, when it is true and when it is false".*

Takeaway: adding a connective involves explaining it in two contexts.

Typical modalities

$\mu, x \vDash^{+} \varphi$ is for "φ is asserted at a world x of model μ".

$\mu, x \vDash^{-} \varphi$ is for "φ is rejected at a world x of model μ".

Validity clauses:

$$(\forall^{+}) \quad \mu, x \vDash^{+} \Box\varphi \iff \forall y\,(xR^{+}_{\forall}y \text{ implies } \mu, y \vDash^{+} \varphi);$$

$$(\exists^{-}) \quad \mu, x \vDash^{-} \Box\varphi \iff \exists y\,(xR^{-}_{\exists}y \text{ and } \mu, y \vDash^{-} \varphi);$$

$$(\exists^{+}) \quad \mu, x \vDash^{+} \Diamond\varphi \iff \exists y\,(xR^{+}_{\exists}y \text{ and } \mu, y \vDash^{+} \varphi);$$

$$(\forall^{-}) \quad \mu, x \vDash^{-} \Diamond\varphi \iff \forall y\,(xR^{-}_{\forall}y \text{ implies } \mu, y \vDash^{-} \varphi).$$

Remark: there are four accessibility relations involved.

Some notes

Typical modal operators can be distinguished by which of four accessibility relation coincide.

Example: three out of four coincide for $\mathsf{BK}^{\mathsf{FS}}$; all four do for BK.

The are four different modal behaviors conflated into two modal operators.

We can consider them as characterizing four different *partially defined* modal operators.

Two conditions of a modality

Consider the *assertion condition*

$$(\forall^+) \quad \mu, x \vDash^+ \Box\varphi \iff \forall y\,(xR^+_\Box y \text{ implies } \mu, y \vDash^+ \varphi);$$

Q. How can we define the *rejection condition* for this operator?

A. Delegate to the valuations.

Now, this is a well-defined connective:

$$(\forall^+) \quad \mu, x \vDash^+ \Box\varphi \iff \forall y\,(xR^+_\Box y \text{ implies } \mu, y \vDash^+ \varphi);$$
$$(\varnothing^-) \quad \mu, x \vDash^- \Box\varphi \iff x \in v^-(\Box\varphi).$$

Four basic modal operators

A $\forall^+$*-operator* has satisfaction clauses

$$(\forall^+) \quad \mu, x \vDash^+ \forall^+\varphi \iff \forall y\,(xR^+_\forall y \text{ implies } \mu, y \vDash^+ \varphi);$$

$$(\varnothing^-) \quad \mu, x \vDash^- \forall^+\varphi \iff x \in v^-(\forall^+\varphi).$$

A $\exists^-$*-operator* has satisfaction clauses

$$(\varnothing^+) \quad \mu, x \vDash^+ \exists^-\varphi \iff x \in v^+(\exists^-\varphi);$$

$$(\exists^-) \quad \mu, x \vDash^- \exists^-\varphi \iff \exists y\,(xR^-_\exists y \text{ and } \mu, y \vDash^- \varphi).$$

Four basic modal operators

A $\exists^{+}$ *-operator* has satisfaction clauses

$$(\exists^{+}) \quad \mu, x \vDash^{+} \exists^{+}\varphi \iff \exists y\,(xR^{+}_{\exists}y \text{ and } \mu, y \vDash^{+} \varphi);$$

$$(\varnothing^{-}) \quad \mu, x \vDash^{-} \exists^{+}\varphi \iff x \in v^{-}(\exists^{+}\varphi).$$

A $\forall^{-}$ *-operator* has satisfaction clauses

$$(\varnothing^{-}) \quad \mu, x \vDash^{+} \forall^{-}\varphi \iff x \in v^{+}(\forall^{-}\varphi);$$

$$(\forall^{-}) \quad \mu, x \vDash^{-} \forall^{-}\varphi \iff \forall y\,(xR^{-}_{\forall}y \text{ implies } \mu, y \vDash^{-} \varphi).$$

Some results

An extension FDE^b of FDE with four basic modalities is characterized.

Can accommodate *full* modalities:

full necessity $\Box$ is a $\forall^+$- and $\exists^-$-operator;

full possibility $\Diamond$ is a $\exists^+$- and $\forall^-$-operator.

A number of non-modal operators is added to it, including

some conditionals: $\rightarrow_i$, $\Rightarrow_i$, $\rightarrow_c$, $\Rightarrow_c$;

bilattice operators: $\otimes$, $\oplus$, F, T, N, B.

Some correspondence theory to express when accessibility relations coincide.

Algebraizable FDE-based modal logics

Algebraizability

W.J. Blok, D. Pigozzi (1989) *Algebraizable logics*

Theorem

L is algebraizable iff
there are *equivalence formulas* $\Delta(p, q)$ and
there are *defining equations* $\delta(p) = \epsilon(p)$ such that:

i) $\vdash_{L} \Delta(\varphi, \varphi)$;

ii) $\Delta(\varphi, \psi) \vdash_{L} \Delta(\psi, \varphi)$;

iii) $\Delta(\varphi, \psi), \Delta(\psi, \chi) \vdash_{L} \Delta(\varphi, \chi)$;

iv) $\bigwedge \Delta(\varphi_i, \psi_i) \vdash_{L} \Delta(f(\varphi_1, \ldots, \varphi_n), f(\psi_1, \ldots, \psi_n))$;

v) $\varphi \dashv\vdash_{L} \Delta(\delta(\varphi), \epsilon(\varphi))$.

The system

For algebraizability we need

i) *some implication:* we start with intuitionistic $\rightarrow$;

ii) *"congruential"* modality $\bigcirc$.

The language is

$$\mathcal{L}^{\bigcirc}_{\rightarrow} := \{\wedge, \vee, \rightarrow, \bigcirc, \sim\}.$$

The system $\mathsf{FDE}^{\bigcirc}_{\rightarrow}$ is obtained by adding to $\mathsf{FDE}_{\rightarrow}$

$$\frac{\varphi \dashv\vdash \psi}{\bigcirc\varphi \dashv\vdash \bigcirc\psi} \qquad \frac{\sim\varphi \dashv\vdash \sim\psi}{\sim\bigcirc\varphi \dashv\vdash \sim\bigcirc\psi}$$

Algebraizability

Theorem

$\mathrm{FDE}^{\bigcirc}_{\rightarrow}$ is algebraizable with *defining equation* $p = p \rightarrow p$ and *equivalence formula* $p \Leftrightarrow q := (p \leftrightarrow q) \wedge ({\sim}p \leftrightarrow {\sim}q)$.

Some options:

i) we can take *classical* or *connexive* implication instead;
ii) can extend with a number of non-modal operators including $\perp$ and bilattice operators;
iii) we can extend $\bigcirc$ to be a *basic modality* or a *full modality*.

Q: how do we get to corresponding algebras?

A: start with *twist-structures*.

Twist-structures

Outline

Suitable for systems with *strong negation* including systems that contain FDE as a subsystem

The name comes from

M. Kracht (1996) *On extensions of intermediate logics by strong negation*

The idea
is to put a *twist* on how operation over the direct square of some algebra are defined.

An early example of the construction is

J.A. Kalman (1959) *Lattices with involution*

Twist-structure

$\mathfrak{A} = \langle A, \wedge, \vee, \to, \bigcirc^+, \bigcirc^- \rangle$ is an *implicative 2op-lattice* if

i) $\langle A, \wedge, \vee, \to \rangle$ is an implicative lattice;

ii) $\bigcirc^+$ and $\bigcirc^-$ are arbitrary unary operations.

A *full twist-structure over* $\mathfrak{A}$ is $\mathfrak{A}^{\bowtie} = \langle A \times A, \wedge, \vee, \to, \bigcirc, \sim \rangle$:

$$
\begin{aligned}
(a, b) \wedge (c, d) &= (a \wedge c, b \vee d); \\
(a, b) \vee (c, d) &= (a \vee c, b \wedge d); \\
(a, b) \to (c, d) &= (a \to c, a \wedge d); \\
\bigcirc(a, b) &= (\bigcirc^+ a, \bigcirc^- b); \\
\sim(a, b) &= (b, a).
\end{aligned}
$$

A *twist-structure over* $\mathfrak{A}$ is a subalgebra $\mathfrak{B}$ of $\mathfrak{A}^{\bowtie}$ such that

$$\pi_1(\mathfrak{B}) = \{a \mid \exists b : (a, b) \in \mathfrak{B}\} = A.$$

Lindenbaum-Tarski with a twist

Put $\mathfrak{A}_{\mathsf{FDE}^{\bigcirc}_{\to}} = \langle A_{\mathsf{FDE}^{\bigcirc}_{\to}}, \wedge, \vee, \to, \bigcirc^{+}, \bigcirc^{-} \rangle$, where

$$
\begin{array}{rcl}
A_{\mathsf{FDE}^{\bigcirc}_{\to}} & := & \{[\varphi] \mid \varphi \in \mathsf{Form}\,\mathcal{L}^{\bigcirc}_{\to}\}; \\
[\varphi] & := & \{\psi \mid \varphi \dashv\vdash_{\mathsf{FDE}^{\bigcirc}_{\to}} \psi\}; \\
[\varphi] * [\psi] & := & [\varphi * \psi], \qquad * \in \{\wedge, \vee, \to\} \\
\bigcirc^{+}[\varphi] & := & [\bigcirc\varphi]; \\
\bigcirc^{-}[\varphi] & := & [\sim \bigcirc \sim\varphi]; .
\end{array}
$$

Then $\mathfrak{A}_{\mathsf{FDE}^{\bigcirc}_{\to}}$ is an implicative 2op-lattice.

Put $\mathfrak{B}_{\mathsf{FDE}^{\bigcirc}_{\to}} = \langle B_{\mathsf{FDE}^{\bigcirc}_{\to}}, \wedge, \vee, \to, \bigcirc, \sim \rangle$, where

$$B_{\mathsf{FDE}^{\bigcirc}_{\to}} = \{([\varphi], [\sim\varphi]) \mid \varphi \in \mathsf{Form}^{\bigcirc}_{\to}\}.$$

Then is $\mathfrak{B}_{\mathsf{FDE}^{\bigcirc}_{\to}}$ a twist-structure over $\mathfrak{A}_{\mathsf{FDE}^{\bigcirc}_{\to}}$.

Completeness

Theorem

$$\Gamma \vdash_{\mathrm{FDE}_{\rightarrow}^{\bigcirc}} \Delta \iff \Gamma \models^{\bowtie}_{\mathrm{FDE}_{\rightarrow}^{\bigcirc}} \Delta \iff \Gamma \models_{\mathfrak{B}_{\mathrm{FDE}_{\rightarrow}^{\bigcirc}}} \Delta,$$

where $\models^{\bowtie}_{\mathrm{FDE}_{\rightarrow}^{\bigcirc}}$ is the consequence relation of the class of all twist-structures.

Remark: this works even if we omit implication.

Moreover, let $\mathfrak{B}$ be a twist-structure over $\mathfrak{A}$. Then

i) if $\bigcirc$ is a $\forall^{+}$-operator in $\mathfrak{B}$, then $\bigcirc^{+}$ is a $\Box$-operator in $\mathfrak{A}$;
ii) if $\bigcirc$ is a $\exists^{+}$-operator in $\mathfrak{B}$, then $\bigcirc^{+}$ is a $\Diamond$-operator in $\mathfrak{A}$;
iii) if $\bigcirc$ is a $\forall^{-}$-operator in $\mathfrak{B}$, then $\bigcirc^{-}$ is a $\Box$-operator in $\mathfrak{A}$;
iv) if $\bigcirc$ is a $\exists^{-}$-operator in $\mathfrak{B}$, then $\bigcirc^{-}$ is a $\Diamond$-operator in $\mathfrak{A}$.

Finally, algebras

Put

$a \preceq b$ iff $(a \to b) \to (a \to b) = a \to b$;
$a \approx b$ iff $a \preceq b$ and $b \preceq a$.

$\mathfrak{A} = \langle A, \wedge, \vee, \to, \bigcirc, \sim \rangle$ is an $\mathrm{FDE}^{\bigcirc}_{\to}$*-lattice* if

i) $\langle A, \wedge, \vee \rangle$ is a distributive lattice;

ii) $\sim(a \vee b) = \sim a \wedge \sim b$, $a = \sim\sim a$, $\sim(a \to b) = a \wedge \sim b$;

iii) $\preceq$ is a preordering on $\mathfrak{A}$;

iv) $\approx$ is a congruence w.r.t. $\wedge$, $\vee$, $\to$, $\bigcirc$ and $\sim \bigcirc \sim$ and

v) $\langle A, \wedge, \vee, \to, \bigcirc, \sim \bigcirc \sim \rangle / \approx$ is an implicative 2op-lattice;

vi) $a \leq b$ iff $a \preceq b$ and $\sim b \preceq \sim a$.

$\mathcal{V}^{\mathrm{FDE}^{\bigcirc}_{\to}}$ is the class of all $\mathrm{FDE}^{\bigcirc}_{\to}$-lattices.

Main results

Theorem

Every $\mathrm{FDE}^{\bigcirc}_{\rightarrow}$-lattice is isomorphic to a twist-structure.

Theorem

$\mathcal{V}^{\mathrm{FDE}^{\bigcirc}_{\rightarrow}}$ forms an equivalent algebraic semantics for $\mathrm{FDE}^{\bigcirc}_{\rightarrow}$ with *defining equation* $p \approx p \rightarrow p$ and *equivalence formula* $p \Leftrightarrow q := (p \leftrightarrow q) \wedge ({\sim}p \leftrightarrow {\sim}q)$.

Theorem

$\mathcal{V}^{\mathrm{FDE}^{\bigcirc}_{\rightarrow}}$ is a variety.

Remark: these results can be expanded in a number of ways.

Neighbourhood semantics

Preliminaries

For a partially ordered set $\langle W, \leq \rangle$ put

$$\text{Up}\,(W) = \{X \mid \forall x\,(x \in X \text{ and } x \leq y \text{ implies } y \in X)\}.$$

A *neighbourhood function* on $\langle W, \leq \rangle$ is $N : W \to 2^{\text{Up}\,(W)}$.

An FDE$^{\bigcirc}_{\rightarrow}$*-n-frame* is $\mathcal{W} = \langle W, \leq, N^{+}_{\bigcirc}, N^{-}_{\bigcirc} \rangle$, where

i) $\langle W, \leq \rangle$ is a partially ordered set;

ii) $N^{+}_{\bigcirc}, N^{-}_{\bigcirc}$ are neighbourhood functions on $\langle W, \leq \rangle$.

An FDE$^{\bigcirc}_{\rightarrow}$*-n-model* is $\mu = \langle \mathcal{W}, v^{+}, v^{-} \rangle$, where

i) $\mathcal{W} = \langle W, \leq, N^{+}_{\bigcirc}, N^{-}_{\bigcirc} \rangle$ is an FDE$^{\bigcirc}_{\rightarrow}$-n-frame;

ii) *valuations* v^{+}, v^{-} map p. variables to elements of Up (W).

Semantic consequence

Define inductively $[\varphi]^+$ and $[\varphi]^-$:

$$[p]^+ = v^+(p);$$
$$[\varphi \wedge \psi]^+ = [\varphi]^+ \cap [\psi]^+;$$
$$[\varphi \vee \psi]^+ = [\varphi]^+ \cup [\psi]^+;$$
$$[\varphi \rightarrow \psi]^+ = \{x \mid \check{x} \cap [\varphi]^+ \subseteq \check{x} \cap [\psi]^+\};$$
$$[{\sim}\varphi]^+ = [\varphi]^-;$$
$$[\bigcirc\varphi]^+ = \{x \mid [\varphi]^+ \in N^+_{\bigcirc}(x)\};$$

$$[p]^- = v^-(p);$$
$$[\varphi \wedge \psi]^- = [\varphi]^- \cup [\psi]^-;$$
$$[\varphi \vee \psi]^- = [\varphi]^- \cap [\psi]^-;$$
$$[\varphi \wedge \psi]^- = [\varphi]^+ \cap [\psi]^-;$$
$$[{\sim}\varphi]^- = [\varphi]^+;$$
$$[\bigcirc\varphi]^- = \{x \mid [\varphi]^- \in N^-_{\bigcirc}(x)\};$$

where $\check{x} := \{y \mid x \leq y\}$.

$\Gamma \vDash_n \Delta$ iff for every FDE$^{\bigcirc}_{\rightarrow}$-n-model μ

$$\bigcup\{[\varphi]^+ \mid \varphi \in \Gamma\} \subseteq \bigcap\{[\psi]^+ \mid \psi \in \Delta\}.$$

Completeness

Theorem

$\Gamma \vdash_{\mathsf{FDE}^{\bigcirc}_{\rightarrow}} \Delta$ iff $\Gamma \models_n \Delta$ for any Γ, Δ.

The method is the usual canonical model method; there is more than one canonical model.

W_L is the set of all prime theories, $[\varphi]_L = \{\Gamma \in W_L \mid \varphi \in \Gamma\}$,

$$lN^+_{\bigcirc}(\Gamma) := \{[\varphi]_L \mid \bigcirc\varphi \in \Gamma\};$$
$$gN^+_{\bigcirc}(\Gamma) := \{X \in Up(W_L) \mid \forall\varphi\,(X = [\varphi]_L \implies \bigcirc\varphi \in \Gamma)\}.$$

$N^+_{\bigcirc}$ is a *canonical neighbourhood function* iff

$$\forall\Gamma \in W_L : \; lN^+_{\bigcirc}(\Gamma) \subseteq N^+_{\bigcirc}(\Gamma) \subseteq gN^+_{\bigcirc}(\Gamma).$$

And similarly for $N^-_{\bigcirc}$.

Correspondence theory

Monotonicity rule $\varphi \vdash \psi / \bigcirc \varphi \vdash \bigcirc \psi$ corresponds to

$$X \in N^{+}_{\bigcirc}(x) \text{ and } X \subseteq Y \implies Y \in N^{+}_{\bigcirc}(x).$$

Monotonicity rule $\sim\varphi \vdash \sim\psi / \sim \bigcirc \varphi \vdash \sim \bigcirc \psi$ corresponds to

$$X \in N^{-}_{\bigcirc}(x) \text{ and } X \subseteq Y \implies Y \in N^{-}_{\bigcirc}(x).$$

Remark: we restrict canonical neighbourhood functions to

$$N^{+}_{\bigcirc}(\Gamma) = \{X \in Up(W_L) \mid \exists Y \subseteq X : \; Y \in IN^{+}_{\bigcirc}(\Gamma)\};$$
$$N^{-}_{\bigcirc}(\Gamma) = \{X \in Up(W_L) \mid \exists Y \subseteq X : \; Y \in IN^{-}_{\bigcirc}(\Gamma)\}.$$

Correspondence theory

Over systems containing *monotonicity rules*:

Axioms of $\forall^+$*-operator* correspond to

1. $\forall x \in W : N^+_{\bigcirc}(x) \neq \varnothing$;
2. $X \in N^+_{\bigcirc}(x)$ and $Y \in N^+_{\bigcirc}(x) \implies X \cap Y \in N^+_{\bigcirc}(X)$.

Axioms of $\exists^+$*-operator* correspond to

1. $\forall x \in W : N^+_{\bigcirc}(x) \neq Up(W)$;
2. $X \cup Y \in N^+_{\bigcirc}(X) \implies X \in N^+_{\bigcirc}(x)$ or $Y \in N^+_{\bigcirc}(x)$.

Correspondence theory

Over systems containing *monotonicity rules*:

Axioms of $\forall^-$*-operator* correspond to

1. $\forall x \in W : N^-_{\bigcirc}(x) \neq \varnothing$;
2. $X \in N^-_{\bigcirc}(x)$ and $Y \in N^-_{\bigcirc}(x) \implies X \cap Y \in N^-_{\bigcirc}(X)$.

Axioms of $\exists^-$*-operator* correspond to

1. $\forall x \in W : N^-_{\bigcirc}(x) \neq Up(W)$;
2. $X \cup Y \in N^-_{\bigcirc}(X) \implies X \in N^-_{\bigcirc}(x)$ or $Y \in N^-_{\bigcirc}(x)$.

Thank you!