Tameness in classes of generalized metric structures: quantale-spaces, fuzzy sets, and sheaves

(Joint work with Michael Lieberman and Jiří Rosický)

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Motivation.

1. Shelah conjectured a eventual categoricity transfer in AECs.
2. Grossberg-VanDieren (2006): Partial answer by assuming tameness and categoricity in a successor cardinal.
3. Boney (2014): Under the existence of a proper class of (almost) strongly compact cardinals, any AEC is tame. Shelah-Vasey (2018): Categoricity transfer theorem dropping categoricity assumption in successor cardinal
4. Boney-Z. (2015): If a proper class of almost strongly compact cardinals exists, any MAEC is **d**-tame.
5. Hirvonen-Hyttinen (2009): Under tame-like assumptions, a categoricity transfer theorem holds for homogeneous MAECs.
6. Z. (2012): A stability transfer theorem for **d**-tame MAECs holds.
7. Lieberman-Rosický (2017): Alternative proof of set-theoretical consistency of **d**-tameness on MAECs by using tools of Accessible Categories.

Lieberman-Rosický-Z.: By using those tools, proving set-theoretical consistency of tameness in more general settings (e.g., $\mathbb{V}$-pseudo metric spaces, $\mathbb{V}$ a cocontinuous quantale).

Why quantales?

- Flagg (1997): Any topological space can be seen as a $\mathbb{V}$-pseudo metric space for a suitable quantale $\mathbb{V}$.
- We can deal with C^*-algebras by using quantales (Borceux-Rosický-van den Bossche).

Cocontinuous quantales.

Definition.

(Commutative) **cocontinuous** *quantale*: $\mathbb{V} = \langle V, +, 0, \leq \rangle$ such that

1. $\langle V, \leq \rangle$ is a cocontinuous (i.e., for any $x \in V$, $x = \bigwedge\{y \in V : x \ll y\}$) complete lattice with bottom 0 and top ∞.
2. $\langle V, +, 0 \rangle$ is a (commutative) monoid.
3. Meets distribute with respect to $+$ in both left and right side.

Examples.

- Classical truth values: $V := \{0, \infty\}$.
- Distances: $V := [0, \infty] \subseteq \mathbb{R} \cup \{\infty\}$.

$\mathbb{V}$-pseudo metric spaces

$\mathbb{V}$*-pseudo metric space*: $\langle M, d\rangle$ provided that $M \neq \varnothing$ and $d : M \times M \to V$ satisfies:

1. (Reflexivity) for all $x \in M$ $d(x, x) = 0$.
2. (Symmetry) for all $x, y \in M$ we have that $d(x, y) = d(y, x)$.
3. (Transitivity) for all $x, y, z \in M$, $d(x, y) \leq d(x, z) + d(z, y)$.

Definition.

Given $\langle M, d\rangle$ a $\mathbb{V}$-pseudo metric space, we say that M is *separated* iff $d(x, y) = 0$ implies that $x = y$ for any $x, y \in M$.

$\mathbb{V}$-pseudo metric spaces

Examples.

1. A $[0,\infty]$-pseudo metric space $\langle M, d\rangle$ yields a distance mapping $d : M \times M \to [0,\infty]$. If d is reflexive, transitive, symmetric and separated, we have that $\langle M, d\rangle$ is a metric space.
2. A discrete metric space is a $\{0,\infty\}$-pseudo metric space.
3. Let $\langle V, +, 0, \leq\rangle$ be a cocontinuous quantale. V itself is a $\mathbb{V}$-space provided with $d(x,y) := (y \dot{-} x) + (x \dot{-} y)$.
4. Let $\langle M, d\rangle$ be a $\mathbb{V}$-pseudo metric space and $1 \leq k < \omega$. Then $d : M^k \times M^k \to V$ defined as $d_k((a_1, \cdots, a_k), (b_1 \cdots, b_k)) := \bigvee\{d(a_i, b_i) : 1 \leq i \leq k\}$ is a $\mathbb{V}$-pseudo metric space.

Pseudo-metric structures (cf. Continuous Logic)

Given a language in the setting of cocontinuous logic L, a $\mathbb{V}$-*pseudo metric structure* based on L is a tuple $\mathcal{M} = \langle M; \square^M : \square \in L \rangle$ defined as follows:

1. If $\square \in L$ is a constant symbol, define $\square^M$ as an element in M.
2. If $\square \in L$ is a relational symbol of arity $1 \leq k < \omega$, define $\square^M : M^k \to V$ as nonexpanding map.
3. If $\square \in L$ is a function symbol of arity $1 \leq k < \omega$, define $\square^M : M^k \to M$ as a nonexpanding map.

V-embeddings

Let $\mathbb{V} = \langle V, +, 0, \leq \rangle$ be a cocontinuous quantale and $\mathcal{M}_1 = \langle M_1, d_1 \rangle$, $\mathcal{M}_2 = \langle M_2, d_2 \rangle$ be $\mathbb{V}$-pseudo metric structures in the same language L. An *L-embedding* $h : \mathcal{M}_1 \to \mathcal{M}_2$ is a mapping $h : M_1 \to M_2$ which preserves the L-structure.

Pseudo-$\mathbb{V}$-Abstract Elementary Classes

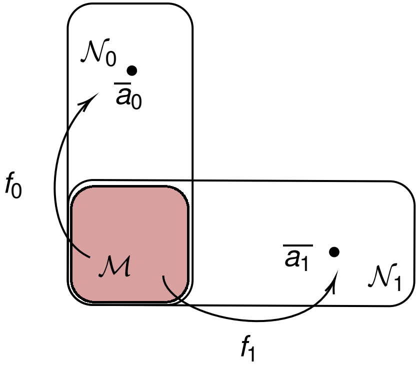
$\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$, $\mathcal{K}$ a class of $\mathbb{V}$-pseudo metric structures in the same language, where $\prec_{\mathcal{K}}$ satisfies:

1. $\prec_{\mathcal{K}}$ is stronger than $\subseteq$.
2. $\mathcal{K}$ is closed under L-isomorphisms.
3. Coherence: if $\mathcal{M}_0 \subseteq \mathcal{M}_1 \prec_{\mathcal{K}} \mathcal{M}_2$ and $\mathcal{M}_0 \prec_{\mathcal{K}} \mathcal{M}_2$ then $\mathcal{M}_0 \prec_{\mathcal{K}} \mathcal{M}_1$.
4. $\mathcal{K}$ is closed under directed colimits.
5. Downward Löhenheim-Skolem.

$\mathcal{K}$-embeddings

L-embeddings $f : \mathcal{M} \to \mathcal{N}$ such that $f[\mathcal{M}] \prec_{\mathcal{K}} \mathcal{N}$.

Galois types $(f_0 : \mathcal{M} \to \mathcal{N}_0, \overline{a}_0) \sim (f_1 : \mathcal{M} \to \mathcal{N}_1, \overline{a}_1)$



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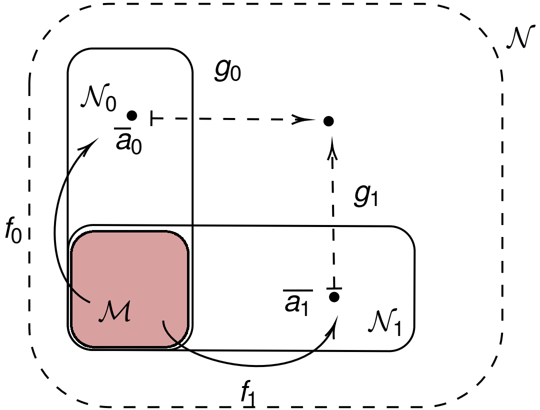
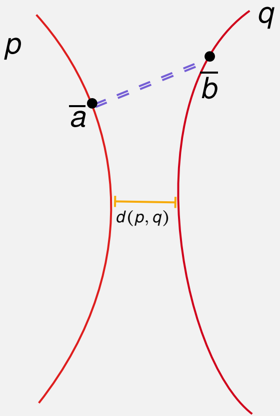
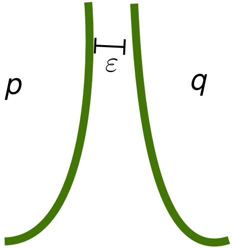


Definition

Given $p, q \in \text{ga-S}_\alpha(M)$, we define $d(p,q) \in V$ as follows:
$d(p,q) := \bigwedge\{d(\overline{a},\overline{b}) : \overline{a} \models p, \overline{b} \models q\} \in V$



$< \kappa$-tameness



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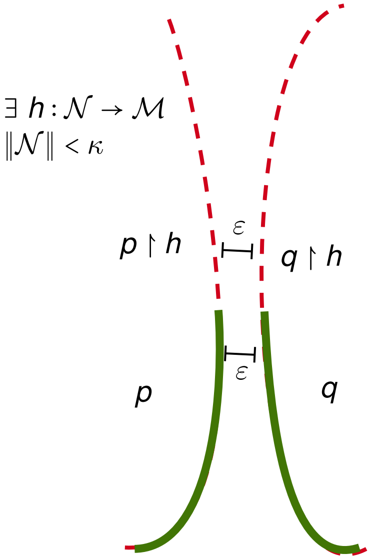

Theorem (Lieberman-Rosický-Z.)

Assuming the existence of a μ-strongly compact cardinal bigger than $|V|^{|V|}$ for any cardinal μ, any pseudo-$\mathbb{V}$-AEC $\mathcal{K}$ is strongly $\mathbb{V}$-tame.

Key points of the proof.

- $\mathcal{L}$: (f_0, f_1, a_0, a_1) -$f_0 : \mathcal{M} \to \mathcal{N}_0$, $f_1 : \mathcal{M} \to \mathcal{N}_1$ and $a_i \in N_i$-.
- $\mathcal{L}_\delta$: (f_0, f_1, a_0, a_1) -$f_0 : \mathcal{M} \to \mathcal{N}_0$, $f_1 : \mathcal{M} \to \mathcal{N}_1$ and $a_i \in N_i$ codifying $d(a_0, a_1) = \delta$-.
- The full image of the forgetful functor $G_\delta : \mathcal{L}_\delta \to \mathcal{L}$ is accessible (by Brooke-Taylor - Rosický).
- Write any (f_0, f_1, a_0, a_1) as a directed colimit of a (cofinal) sequence of restrictions to small $\prec_{\mathcal{K}}$-structures (which by hypothesis are close enough)
- By the accessibility of the full image of G_δ's, (f_0, f_1, a_0, a_1) belongs to the full image of a suitable G_δ, and so the respective Galois-types ga-tp(a_0/f_0), ga-tp(a_1/f_1) are close enough.

More general settings

Definition (partial $\mathbb{V}$-metric spaces)

Given $\mathbb{V} = \langle V, +, 0, \leq \rangle$ a cocontinuous quantale, a *partial* $\mathbb{V}$*-metric space* is a pair $\langle M, d \rangle$ provided that M is a set and $d : M \times M \to V$ satisfies the following properties:

1. (Equality) for all $x, y \in M$ $d(x,x) = d(y,y) = d(x,y)$ implies $x = y$.
2. (Symmetry) for all $x, y \in M$ we have that $d(x,y) = d(y,x)$.
3. (Transitivity) for all $x, y, z \in M$, $d(x,y) + d(z,z) \leq d(x,z) + d(z,y)$.
4. (Small self-distances) $d(x,x) \leq d(x,y)$.

Remark

Why partial V-metric spaces?

1. Partial V-metric spaces allow to defined analogous to Ω-fuzzy sets (Ω a -complete- Heytting algebra).
2. Sheaves on Ω carry a definition of a fuzzy-equality notion on Ω.

Model-theoretic forcing.

- Model-theoretical forcing in metric spaces: Ben-Yaacov and Iovino (2009) -omitting types theorem and separable quotient problem in Banach spaces-.
- Model-theoretical forcing in Booolean and Heytting valued structures.

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Ongoing project (Moncayo-Z.)

- Extending Ben-Yaacov - Iovino work to pseudo-$\mathbb{V}$-metric spaces.
- Understanding model-theoretic forcing on quantale-valued structures and compare it with the BY-I's work.

References

[BeIo09] I. Ben-Yaacov and J. Iovino, *Model theoretic forcing in analysis*, APAL 158 pp 163–174, 2009.

[Fl97] R. Flagg, *Quantales and continuity spaces*, Alg. Univ. 37 pp. 257-276, 1997.

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Děkuji!

Old town Prague - view from Vrtba Garden.