# Elementary theories of PAC structures via Galois groups

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- Cherlin, van den Dries, and Macintyre introduced a notion of the **complete system** of Galois group $G(K)$ of a field $K$, denoted by $SG(K)$.
- For PAC fields $E \subset F$, they show that

$$E \prec F \Leftrightarrow SG(E) \prec SG(F).$$

- Chatzidakis gave a description of complete types in PAC fields in terms of complete systems.
- Chatzidakis($n \geq 3$) and Ramsey($n = 1, 2$) showed that for a PAC field $E$,

$$Th(SG(E)) \models NSOP_n \Rightarrow Th(E) \models NSOP_n.$$

- We aim to generalize previous results for PAC fields to **PAC structures** in term of the **sorted complete system** of Galois group $G(K)$ of a PAC structure $K$, denoted by $SG(K)$.
- Our main results(under certain assumptions) are as follows:
  1. For PAC structures $E \subset F$,
  $$E \prec F \Leftrightarrow SG(E) \prec SG(F).$$
  2. Description of complete types in PAC structures in terms of sorted complete systems.
  3. For a PAC structure $E$, and $n \geq 1$,
  $$Th(SG(E)) \models NSOP_n \Rightarrow Th(E) \models NSOP_n.$$

- Let $\mathcal{L}$ be a first order language with a fixed set $\mathcal{S}$ of sorts.
- For $J := (S_1, \dots, S_n) \in \mathcal{S}^{<\omega}$, put $S_J := S_1 \times \cdots \times S_n$.
- Let $T$ be a complete stable theory in $\mathcal{L}$, having
  1. QE; and
  2. the uniform EI, that is, for any $\emptyset$-definable equivalence relation $E$ on $S_J$, there is an $\emptyset$-definable function $f_E : S_J \to S_{J'}$ for some $J' \in \mathcal{S}^{<\omega}$, whose fibers are $E$-classes, such that

$$T \models \forall x, y \in S_J (f_E(x) = f_E(y) \leftrightarrow E(x, y)).$$

- Let $\mathfrak{C} \models T$ be a fixed monster model.
- We denote $K, E, F, \dots$ for **definably closed** small substructure of $\mathfrak{C}$, and denote $M, N, \dots$ for elementary substructure of $\mathfrak{C}$.

- The **(Shelah-)Galois group** of $K$ is

$$G(K) = \{\sigma \restriction_{\mathrm{acl}(K)} \mid \sigma \in \mathrm{Aut}(\mathfrak{C}/K)\}.$$

- Let $K \subset E \subset \bar{K}(:= \mathrm{acl}(K))$. We say that $E$ is normal if for any $\sigma \in \mathrm{Aut}(\mathfrak{C}/K)$, $\sigma[E] = E$.
- For a normal extension $E$ of $K$, put

$$G(E/K) := \{\sigma \restriction_E \mid \sigma \in G(K)\}.$$

- For a normal extension $E$ of $K$ and $J \in \mathcal{S}^{<\omega}$, put

$$\mathrm{Aut}_J(E/K) := \{\sigma \restriction_{S_J(E)} \mid \sigma \in G(K)\}.$$

Then, we have a natural epimorphism

$$\mathrm{res} : G(E/K) \to \mathrm{Aut}_J(E/K), \sigma \mapsto \sigma \restriction_{S_J(E)} .$$

- Let $K \subset E$. We say that $E$ is a **regular** extension of $K$ if $K$ is algebraically closed in $E$, that is,

$$\operatorname{acl}(K) \cap E = K.$$

- Let $K \subset M \prec \mathfrak{C}$. We say that $K$ is **pseudo-algebraically closed(PAC)** in $M$ if for any regular extension $K \subset E \subset M$, $K$ is existentially closed in $E$.

**Fact**

*$K$ is PAC in $\mathfrak{C}$ if and only if any stationary type over $K$ is finitely satisfiable in $K$.*

The **sorted complete system** $SG(K)$ of Galois group $G(K)$ is the following multi-sorted structure:

Underlying set:

- For an open normal subgroup $N$ of $G(K)$ with $L := \bar{K}^N$, the set

$$\mathcal{F}(N) := \{J \in \mathcal{S}^{<\omega} : \ \mathrm{res} : G(L/K) \cong \mathrm{Aut}_J(L/K)\}$$

  is called **the sorting data** of $N$.
- For $(k, J) \in \omega \times \mathcal{S}^{<\omega}$, the sort

$$m(k, J) := \{(\sigma N, k, J) |\ \sigma \in G(K), N \triangleleft_{open} G(K), \}$$

  such that $[G(K) : N] \leq k$ and $J \in \mathcal{F}(N)$.

**Theorem (meaning of sorting data)**

*Let $N \triangleleft_{open} G(K)$ with $[G(K) : N] \leq k$, and let $J \in \mathcal{S}^{<\omega}$. Put $L := \bar{K}^N$. The following are equivalent:*

1. *$J \in \mathcal{F}(N)$.*
2. *There is a tuple $a \in S_J(L)^k$ such that $\mathrm{dcl}(Ka) = L$.*

Fix $(k,J),(k',J') \in \omega \times \mathcal{S}^{<\omega}$.

- A binary relation, $\leq (:=\leq_{k,k',J,J'}) \subset m(k,J) \times m(k',J')$: For $\sigma N \in m(k,J)$ and $\sigma' N' \in m(k',J')$,
$$\leq (\sigma N, \sigma' N') \text{ if and only if } k \geq k' \text{ and } N \subset N'$$
,that is, there is a natural epimorphism $\pi : G(K)/N \to G(K)/N'$.
- A binary relation, $C(:= C_{k,k',J,J'}) \subset m(k,J) \times m(k',J')$: For $\sigma N \in m(k,J)$ and $\sigma' N' \in m(k',J')$,
$$C(\sigma N, \sigma' N') \text{ if and only if } k \geq k' \text{ and } \sigma N \subset \sigma' N'$$
,that is, $\pi(\sigma N) = \sigma' N'$.
- A ternary relation, $P(:= P_{k,J}) \subset m(k,J)^3$: For $\sigma_1 N_1, \sigma_2 N_2, \sigma_3 N_3 \in m(k,J)$,
$$P(\sigma_1 N_1, \sigma_2 N_2, \sigma_3 N_3) \text{ if and only if } N_1 = N_2 = N_3 (=: N) \text{ and } \sigma_1 \sigma_2 N = \sigma_3 N.$$
, that is, $P$ encodes the graphs of the group operations of the quotient groups $G(K)/N$.

- The sorted complete system generalizes the complete system of Galois group of field.
- The sorted complete systems are axiomatizable in the language $\mathcal{L}_G(\mathcal{S}) := \{\leq_{k,k',J,J'}\} \cup \{C_{k,k',J,J'}\} \cup \{P_{k,J}\}$ with a set of sorts, $\omega \times \mathcal{S}^{<\omega}$.

**Theorem**

*(T:not necessarily stable) Let $M \prec \mathfrak{C}$ and $K \subset M$. Then $SG(K)$ is interpretable in the $\mathcal{L}_P(:= \mathcal{L} \cup \{P\})$-structure $(M, K)$. And the interpretation of $SG(K)$ does not depend on the choice of $M$.*

Interpretability of the sorted complete system

- The set of primitive elements of finite normal extensions of $K$ is definable in the $\mathcal{L}_P$-structure $(M, K)$.

## Definition

Let $n \geq 1$ and let $J \in \mathcal{S}^{<\omega}$. Let $a_1, \ldots, a_n \in S_J$. We say $a_1, \ldots, a_n$ are *conjugated over* $K$ if

- $\ulcorner\{a_1, \ldots, a_n\}\urcorner \in K$; and
- $\ulcorner A \urcorner \notin K$ for a proper subset $A$ of $\{a_1, \ldots, a_n\}$,

denoted by $\mathrm{Conj}^n_{J,K}(a_1, \ldots, a_n)$. If $J$, $n$, and $K$ are obvious, we omit them.

- Note that $\mathrm{Conj}^n_{J,K}(a_1, \ldots, a_n)$ if and only if $a_1 \in \mathrm{acl}(K)$, $\mathrm{Gal}(K)a_1 = \{a_1, \ldots, a_n\}$, and $|\mathrm{Gal}(K)a_1| = n$.

## Definition

Let $n \geq 1$ and let $J \in \mathcal{S}^{<\omega}$. We say $a \in S_J$ is *an n-primitive element over K* if there are $a_2, \ldots, a_n \in S_J$ with $\alpha := (a, a_2, \ldots, a_n)$ such that

1. $\mathrm{Conj}(a, a_2, \ldots, a_n)$; and
2. $\mathrm{Conj}(\alpha, \alpha_2, \ldots, \alpha_n)$ for some $\alpha_2, \ldots, \alpha_n \in S_J^n$.

- Using the relation between sorting data and primitive elements, we interpret $SG(K)$ in $(M, K)$.

We assume some additional assumptions, which hold for $ACF_p$ and $DCF_0$:

- $T$ has non finite covering property(nfcp).
- PAC property is of first order property: There is a set $\Sigma$ of $\mathcal{L}_P$-formulas such that

$$(M, P) \models T \cup \Sigma \Leftrightarrow M \models T \wedge P \text{ is PAC in } M.$$

- $T$ has the boundary property $B(3)$: For any small subset $A$ of $\mathfrak{C}$, and any $A$-independent set $\{a_0, a_1, a_2\}$, we have that

$$\operatorname{dcl}(\operatorname{acl}(Aa_0a_2)\operatorname{acl}(Aa_1a_2)) \cap \operatorname{acl}(Aa_0a_1) = \operatorname{dcl}(\operatorname{acl}(Aa_0)\operatorname{acl}(Aa_1)).$$

**Lemma**

*Let $T$ be a stable theory with QE and EI. Suppose $T$ has* **nfcp**. *Let $\mathfrak{C} \models T$ be a monster model. Let $E \subset F \subset \mathfrak{C}$ be small substructures such that $E \prec F$. Then there are $M \prec N \prec \mathfrak{C}$ such that*

$$(M, E) \prec (N, F)$$

*as $\mathcal{L}_P$-structures.*

**Theorem**

*Let $E \subset F$ be PAC substructures in $\mathfrak{C}$. Then, we have that*

$$E \prec F \Leftrightarrow SG(E) \prec SG(F).$$

## Theorem (Description of complete types in PAC structures)

*Let $F$ be PAC in $\mathfrak{C}$. Let $a, b \in F$ be tuples(of possibly infinite length) and $C \subset F$. Put $A := \mathrm{acl}_F^r(aC)$ and $B := \mathrm{acl}_F^r(bC)$, where $\mathrm{acl}_F^r(D) := \mathrm{acl}(D) \cap K$ for $D \subset F$. The followings are equivalent:*

1. $\mathrm{tp}_F(a/C) = \mathrm{tp}_F(b/C)$.
2. *There is an $\mathcal{L}(C)$-embedding $\phi : \mathrm{acl}(A) \to \mathrm{acl}(B)$ such that*
    1. $\phi(a) = b$ *and* $\phi[A] = B$*; and*
    2. $S\Phi : SG(A) \to SG(B)$ *is a partial elementary map of* $SG(F)$*, where the automorphism* $\phi : \mathrm{acl}(A) \to \mathrm{acl}(B)$ *induces an isomorphism* $\Phi : G(B) \to G(A), \tau \mapsto \phi^{-1} \circ \tau \circ \phi$*, and* $\Phi$ *induces an isomorphism* $S\Phi : SG(A) \to SG(B), \sigma N \mapsto \Phi^{-1}(\sigma)\Phi^{-1}[N]$.

**Theorem (Generalized $NSOP_n$ criteria of Z. Chatzidakis and N. Ramsey)**

*Let $F$ is PAC in $\mathfrak{C}$ and let $n \geq 1$.*

$$Th(SG(F)) \models NSOP_n \Rightarrow Th(F) \models NSOP_n$$

- In the proof of $NSOP_n$ criteria for PAC fields, Chatzidakis' type-amalgamation theorem for PAC fields is a crucial ingredient.
- We generalize Chatzidakis' type-amalgamation theorem to PAC structures(here is the only place where the assumption of $B(3)$ is used), and using this, we generalize $NSOP_n$ criteria to PAC structures.

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Thank you!