On universal modules with pure embeddings

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Abstract elementary classes

Abstract elementary classes (AECs) were introduced by Shelah in the 1980s to unify various *ad hoc* methods in non-elementary model theory (mostly infinitary logic) where important consequences of the Compactness Theorem were not available.

Abstract elementary classes—limit models

Limit models—limits of universal models—were introduced by Kolman and Shelah as a substitute in AECs for saturated models.

Abstract elementary classes—Galois types

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- (hereditary) invariance under automorphisms

“Galois types”, “semantic types”

Abstract elementary classes—Galois stability

Definition

An AEC **K** is λ-Galois stable

iff

for every $\mathcal{M}$ of cardinality $\leq \lambda$,
the set of Galois types over $\mathcal{M}$ also has cardinality $\leq \lambda$.

Modules

Context

- R is a ring with unit;
- $\mathcal{L}_R$ is the usual language for (left) R-modules;
- $\tau = |\mathcal{L}_R|$;
- T is some consistent theory extending the usual theory of (left) R-modules.

$\mathbf{K}^T$ is the abstract class consisting of all models of T with the order given by "pure substructure".

Modules: AEC

Lemma

$\mathbf{K}^T$ *is an AEC*

- *with Löwenheim-Skolem number* τ *,*
- *and no maximal models.*

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Lemma

If $\mathbf{K}^T$ *is closed under direct sums,*
then $\mathbf{K}^T$ *has*

- *the joint embedding property*
  and
- *the amalgamation property.*

Modules: interesting AECs

- torsion free abelian groups;
- the class of models of any complete theory of modules;
- the class of all R-modules;
- any universal Horn class of modules;
- the torsion-free class of a radical of finite type;
- the torsion class of a left exact radical closed under products;
- the class of flat left modules over a right coherent ring;
- …

$\mathbf{K}^T$: main hypothesis

Main Hypothesis

$\mathbf{K}^T$ has

- an infinite model;
- joint embedding;
- the amalgamation property.

$\mathbf{K}^T$: Galois stability

Theorem (Theorem 3.16)

If $\lambda = \lambda^\tau$, then $\mathbf{K}^T$ is λ-Galois stable.

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Note that every complete theory of modules is stable.

$\mathbf{K}^T$: Universal models

Theorem (Theorem 3.19)

If

$$\lambda = \lambda^\tau$$

or

$$\forall \mu < \lambda (\mu^\tau < \lambda)$$

Then $\mathbf{K}^T$ *has a universal model of cardinality* λ *.*

Generalization

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Corollary (Cor. 3.20)

Let **K** *be an AEC satisfying*

- *joint embedding, amalgamation, no maximal models*
- *There is* $\theta_0 \geq \mathrm{LS}(\mathbf{K})$ *and* κ *such that for all* $\theta \geq \theta_0$ *, if* $\theta^\kappa = \theta$ *, then* **K** *is* θ*-Galois stable;*

Then:

- *For* $\lambda > \theta_0$ *, if* $\lambda^\kappa = \lambda$ *or* $\forall \mu < \lambda (\mu^\kappa < \lambda)$ *, then there is a universal model of cardinality* λ *.*

$\mathbf{K}^T$: Limit models

Corollary (Cor. 4.1)

$\mathbf{K}^T$ *has a* (λ, α)*-limit model whenever* $\lambda^\tau = \lambda$ *and* $\alpha < \lambda^+$ *.*

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Lemma (Lemma 4.3)

If

$\mathcal{A}, \mathcal{B} \in \mathbf{K}^T$ *are limit models*

then

$\mathcal{A} \equiv \mathcal{B}$.

$\mathbf{K}^T$: "Big" limit models

Theorem (Theorem 4.4)

Suppose $\lambda \geq \tau^+$ *.*
If $\mathcal{M}$ *is a* (λ, α)*-limit model with* $\mathrm{cf}(\alpha) \geq \tau^+$ *, then* $\mathcal{M}$ *is pure-injective.*

$\mathbf{K}^T$: "Big" limit models

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As a consequence of Bumby's theorem,

Corollary (Corollary 4.6)

Suppose $\lambda \geq \tau^+$ *.*
If

$\mathcal{M}$ *and* $\mathcal{M}'$ *are* λ*-limit models of cofinality* $\geq \tau^+$

then

$\mathcal{M} \cong \mathcal{M}'$ *.*

$\mathbf{K}^T$: "Small" limit models

Assume in addition that $\mathbf{K}^T$ is closed under direct sums.

Theorem (Theorem 4.8)

Suppose $\lambda \geq \tau^+$.

If

- $\mathcal{M}$ *is a* (λ, ω)*-limit model and*
- $\mathcal{N}$ *is a* (λ, τ^+)*-limit model*

then

- $\mathcal{M} \cong \mathcal{N}^{(\aleph_0)}$.

$\mathbf{K}^T$: limit models: spectrum

Theorem (Theorem 4.12)

Let R be countable, and T a theory of R-modules satisfying the Main Hypothesis.

1. *If $\mathbf{K}^T$ is Galois-superstable, then there is $\mu < \beth_{(2^{\aleph_0})^+}$ such that for every $\lambda \geq \mu$ there is a unique limit model of cardinality λ .*

$\mathbf{K}^T$: limit models: spectrum

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1. *If $\mathbf{K}^T$ is Galois-superstable, then there is $\mu < \beth_{(2^{\aleph_0})^+}$ such that for every $\lambda \geq \mu$ there is a unique limit model of cardinality λ.*
2. *If $\mathbf{K}^T$ is not Galois-superstable, then for every $\lambda \geq \aleph_1$, there are either no limit models of cardinality λ, or there are precisely two limit models of cardinality λ.*

Torsion-free abelian groups

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$\mathbf{K}^{tf}$ is the AEC of torsion-free abelian groups with pure embeddings.

Theorem (Mazari-Armida)

If $\mathcal{A} \in \mathbf{K}^{\mathrm{tf}}$ is a (λ, α)-limit model with $\mathrm{cf}\,\alpha \geq \aleph_1$ then

$$\mathcal{A} \cong \mathbb{Q}^{(\lambda)} \oplus \prod_{p \text{ prime}} \overline{\mathbb{Z}_{(p)}^{(\lambda)}}$$

These are all pure-injective.

Theorem (Theorem 4.14)

If $A \in \mathbf{K}^{tf}$ is a (λ, α)-limit model with $\mathrm{cf}(\alpha) = \omega$, then

$$A \cong \mathbb{Q}^{(\lambda)} \oplus \prod_p \overline{\mathbb{Z}_{(p)}^{(\lambda)}}^{(\aleph_0)}.$$

Hence no such limit model is pure-injective.

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Observation

In the AEC of torsion-free abelian groups the existence of a universal model of cardinality $\aleph_n$ is independent of ZFC, for any $1 \leq n \leq \omega$.

Future work. . .

- Examples and counterexamples in AECs of modules, based on the current understanding of the finitary first order theory of modules.
- Applications of the technology of AECs to non-elementary classes of modules.

Some references

- Marcos Mazari-Armida: Algebraic descriptions of limit models in classes of abelian groups, arXiv:1810.02203
- Thomas G. Kucera and Marcos Mazari-Armida: On universal modules with pure embeddings, arXiv:1903.00414
- Marcos Mazari-Armida: Superstability, noetherian rings and pure-semisimple rings, arXiv:1908.02189.