

On Automorphism Groups of Hrushovski's Pseudoplanes in Rational Cases

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Logic Colloquium 2019, Prague
August 16, 2019

Supported by JSPS KAKENHI Grant No. 17K05345.

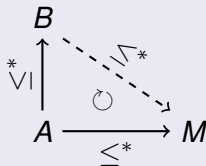
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Definition

Let \mathbf{K} be a class of finite (finitely generated) structures. Suppose that there is a relation \leq^* between structures in \mathbf{K} . We say that A is closed in B if $A \leq^* B$. (e.g. $A \leq^* B$ if A is a substructure of B .)

A countable \mathcal{L} -structure M is called a **Fraïssé Limit** (or a **generic structure**) of (\mathbf{K}, \leq^*) if

- $A \subseteq_{\text{fin}} M \Rightarrow$ there is B such that $A \subseteq B \subseteq_{\text{fin}} M$ and $B \leq^* M$;
- $A \subset_{\text{fin}} M \Rightarrow A \in \mathbf{K}$;
- for any A, B in \mathbf{K} with $A \leq^* B$ and $A \leq^* M$,



Fact

(\mathbf{K}, \leq^*) has a Fraïssé limit if and only if (\mathbf{K}, \leq^*) has HP (the hereditary property), JEP (the joint embedding property), and AP (the amalgamation property).

We omit the definitions.

Fact

Let M be a Fraïssé limit of (\mathbf{K}, \leq^*) .

(1) M is unique up to isomorphisms.

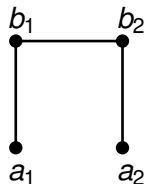
(2) Suppose $A \leq^* M$ and $B \leq^* M$ and $\sigma_0 : A \rightarrow B$ is an isomorphism where A, B are finite. Then σ_0 can be extended to an automorphism of M .

Predimension Function

We want to define a new closedness relation between graphs.
Let A be a graph. Put

$$\delta_\alpha(A) = |A| - \alpha e(A)$$

where α is a real number with $0 < \alpha < 1$,
 $|A|$ is the number of the vertices of A , and
 $e(A)$ is the number of the edges of A .



For this graph,

$$\delta_{2/3} = 4 - 3 \times (2/3) = 2, \text{ and}$$

$$\delta_{5/8} = 4 - 3 \times (5/8) = 2 + (1/8).$$

Closed Substructures

Suppose $A \subseteq_{\text{fin}} B$ (induced subgraph).

$A \leq_{\alpha} B$ if

$$A \subseteq X \subseteq_{\text{fin}} B \Rightarrow \delta_{\alpha}(A) \leq \delta_{\alpha}(X).$$

$A <_{\alpha} B$ if

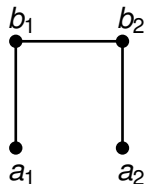
$$A \subsetneq X \subseteq_{\text{fin}} B \Rightarrow \delta_{\alpha}(A) < \delta_{\alpha}(X).$$

In the rest of this talk, A is **closed in** B if $A <_{\alpha} B$.

Put

$$\mathbf{K}_{\alpha} = \{A : \text{finite} \mid \emptyset <_{\alpha} A\}.$$

We often omit subscript α .



Suppose $\alpha = 5/8$.

$$\delta_{5/8} = 4 - 3 \times (5/8) = 2 + (1/8).$$

$$a_1 a_2 < a_1 a_2 b_1 b_2$$

Closure and Dimension

Let M be a Fraïssé limit of an amalgamation class $(\mathbf{K}, <)$.

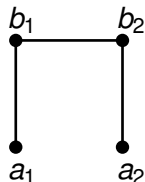
For a finite $A \subseteq M$, there is a smallest B satisfying $A \subseteq B < M$, called a closure of A , written $\text{cl}(A)$.

For a finite $A \subseteq M$, put $d(A) = \delta(\text{cl}(A))$.

Put $d(A/B) = d(AB) - d(B)$.

$B \perp_A C$ iff $d(B/AC) = d(B/A)$.

(Assume $A \subseteq C$) $b \perp_A C$ iff $b \perp_A C$ and $\text{cl}(bC) = \text{cl}(bA) \cup C$.



Suppose $\alpha = 5/8$.

Suppose this graph is closed in M .

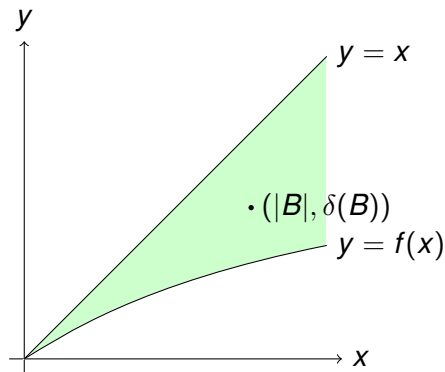
$\text{cl}(b_1 a_1 a_2) = b_1 b_2 a_1 a_2$.

$d(b_1/a_1 a_2) = 1/8$.

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a log-like function.

e.g. $\frac{1}{2} \log_3(x+1)$, $\frac{1}{2} \log_2(x+2) - 1$

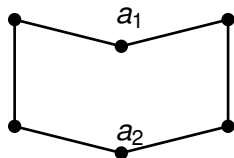
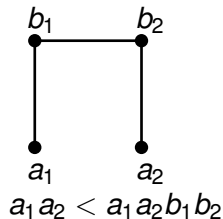
$$K_f = \{A : \text{finite} \mid B \subseteq A \Rightarrow \delta(B) \geq f(|B|)\}$$



Fact

Assume that $(\mathbf{K}_f, <)$ has the free amalgamation property.
 i.e. If $A, B, C \in \mathbf{K}_f$ with $A < B$ and $A < C$ then $B \otimes_A C \in \mathbf{K}_f$. Here,
 $B \otimes_A C$ is a graph such that $V(B \otimes_A C) = V(B) \cup V(C)$ with
 $V(B) \cap V(C) = V(A)$, and $E(B \otimes_A C) = E(B) \cup E(C)$.
 A Fraïssé limit M of $(\mathbf{K}_f, <)$ exists in this case.
 If f is unbounded, $Th(M)$ is \aleph_0 -categorical.

Suppose $\alpha = 5/8$.



An amalgam of copies over $a_1 a_2$

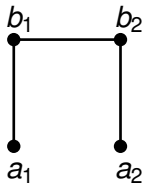
(The following notions are due to Evans, Ghadernezhad, Tent.)

Let M be a Fraïssé limit of \mathbf{K}_f .

Let $A \subseteq M$ be finite and $b \in M$ (a single element). We say that b is *basic* over A if $b \notin A$ and whenever $A \subseteq C < M$ and $d(b/C) < d(b/A)$ then $b \in C$. In this case, the orbit of b over A is called a *basic orbit* over A .

If $\delta(b/A)$ is the smallest possible positive dimension, then b is basic over A .

If b is basic over A , $\text{tp}(b/A)$ acts like a regular type.



Let $\alpha = 5/8$. We have $2/3 - 5/8 = 1/8$.

$d(b_1/a_1 a_2) = 1/8$.

b_1 is basic over $a_1 a_2$.

Theorem (Evans, Ghadernezhad, Tent)

Let α be a rational number with $0 < \alpha < 1$.

Suppose \mathbf{K}_f has the FAP with respect to $<_\alpha$ and $\text{cl}(\emptyset) = \emptyset$.

Let M be the Fraïssé limit of $(K_f, <_\alpha)$.

If $M = \text{cl}(A, D)$ for some basic orbit D over $A \subset M$

then $\text{Aut}(M)$ is a simple group.

Previous to their work, there are works by Truss, Lascar, Macpherson, Tent, and Ziegler. The proof uses some facts from the descriptive set theory about Polish groups.

Theorem (Evans, Ghadernezhad, Tent)

Assume that

$0 < \alpha = c/d < 1$ with coprime positive integers c and d ,

$f(0) = 0, f(1) = 1,$

$f'_+(x) \leq 1/(d \cdot x)$ where d is the denominator of α , and

$f'_+(x)$ is non-increasing.

Let M be the Fraïssé limit of $(K_f, <_\alpha)$.

If $\alpha = 1$ with R a ternary relation, then $M = \text{acl}(A, D)$ for some basic orbit D over some A of M .

If $\alpha = 1/2$ with R a binary relation, then $M = \text{acl}(A, D)$ for some basic orbit D over some A of M .

Therefore, $\text{Aut}(M)$ is a simple group in these cases.

They conjectured that $\text{Aut}(M)$ is a simple group if α is any rational number with $0 < \alpha < 1$.

Hrushovski's Boundary Function

Let α be a positive real number. We define x_n, e_n, k_n, d_n for integers $n \geq 1$ by induction as follows:

Let $x_0 = 0, e_0 = 0, x_1 = 2$ and $e_1 = 1$.

Assume that x_n and e_n are defined.

Let r_n be the smallest rational number r such that $r = k/d > \alpha$ for some positive integers k, d with $d \leq e_n$.

Let k_n and d_n be coprime positive integers with $k_n/d_n = r_n$.

Finally, let $x_{n+1} = x_n + k_n$, and $e_{n+1} = e_n + d_n$.

Let f be a continuous function from \mathbb{R}^+ to \mathbb{R}^+ satisfying the following:

- $f(x_n) = x_n - \alpha e_n$ for each $n \geq 0$.
- f is linear on $[x_n, x_{n+1}]$ for each $n \geq 0$.

We call such f a *Hrushovski's boundary function associated to α* .

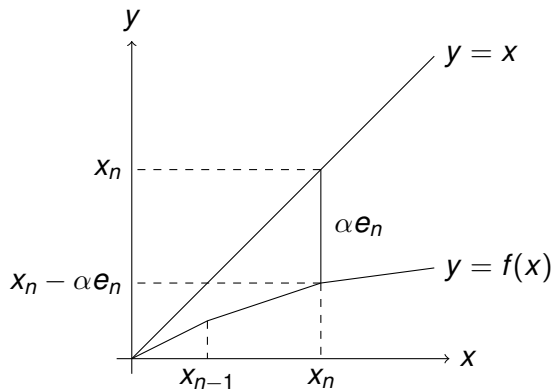
(x_n, e_n) has the following meaning:

e_n is the maximum number of edges allowed among x_n vertices.

$x_1 = 2$ and $e_1 = 1$ mean that only 1 edge is allowed on 2 vertices.

Hrushovski's Boundary Functions

Hrushovski's boundary functions are piecewise linear functions.



Example with $\alpha = 5/8$

n	1	2	3	4	5	6	7	8	9
x_n	2	3	4	6	8	10	17	29	51
e_n	1	2	3	6	9	12	23	42	77
k_n	1	1	2	2	2	7	12	22	47
d_n	1	1	3	3	3	11	19	35	75

For $n \geq 3$, $k_n = 2 + 5\ell$ and $d_n = 3 + 8\ell$ for some ℓ .

Proposition

Suppose $0 < \alpha = c/d < 1$ with coprime positive integers c and d .

Let a, b be positive integers with

$$ad - bc = 1, \quad a \leq c, \quad \text{and} \quad b < d.$$

Eventually, e_n, k_n , and d_n satisfy the following:

Given e_n , let ℓ be the largest integer ℓ' with $b + \ell'd \leq e_n$.

Then $k_n = a + \ell c$ and $d_n = b + \ell d$.

In particular, $dk_n - cd_n = 1$.

Let f be a Hrushovski's boundary function.

$(\mathbf{K}_f, <)$ has the FAP, and it has the Fraïssé limit.

The right derivative $f'_+(x)$ tends to 0 as $x \rightarrow \infty$.

If α is rational, then f is unbounded. In this case, the Fraïssé limit of \mathbf{K}_f is supersimple.

If α is irrational, f can be unbounded or bounded. In this case, the Fraïssé limit of \mathbf{K}_f is strictly stable. f is bounded for “usual” irrationals like $1/\sqrt{2}$. There are irrational numbers α such that f is unbounded by Baire Category Theorem.

Theorem (K.)

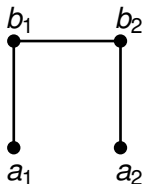
Suppose α is rational with $0 < \alpha < 1$. Let f be the Hrushovski's boundary function with respect to α . Let M be the Fraïssé limit of $(K_f, <)$.

Then there is a basic orbit D over some $A \subset M$ such that $M = \text{cl}(A, D)$. Therefore, $\text{Aut}(M)$ is a simple group.

We explain a sketch of the proof in the case $\alpha = 5/8$.

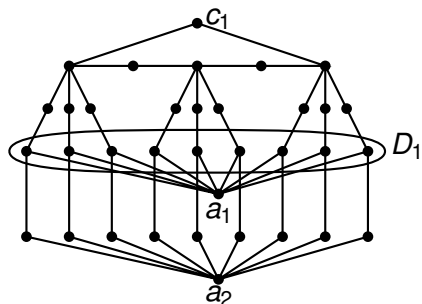
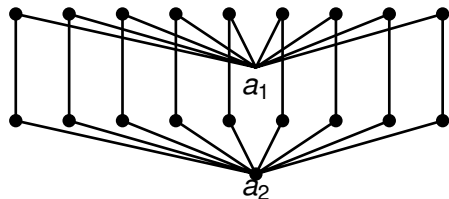
The following graph is embedded into M as a closed set.

Since $d(b_1/a_1 a_2) = 1/8$, b_1 is basic over $a_1 a_2$.



Main Theorem

By the FAP, the following graph belongs to K_f :



This graph belongs to K_f also,
and $d(c_1/D_1) = 0$ in this graph.
Embedded in M as a closed set,
 D_1 is a subset of the basic orbit
over A , and
 $c_1 \in \text{cl}(D_1)$.

By repeating the similar argument, we can make “the tower” bigger. Eventually, we get the following claim:

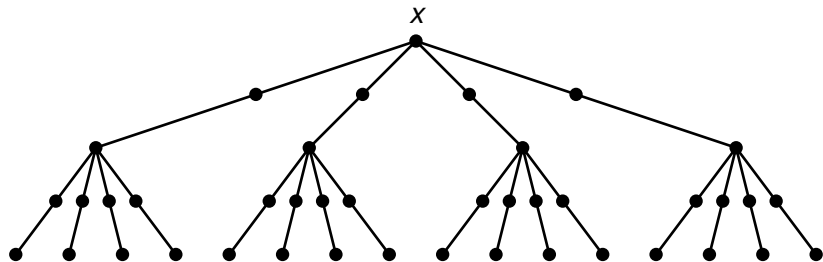
Claim 1. Let D be the orbit of b_1 over $a_1 a_2$. Then there is $c \in \text{cl}(a_1 a_2, D)$ such that $c \perp a_1 a_2$.

Therefore, $\{y \in M \mid y \perp a_1 a_2\} \subseteq \text{cl}(a_1 a_2, D)$.

Claim 2. $x \in \text{cl}(a_1 a_2, D)$ for any $x \in M$.

Let $x \in M$ and $X = \text{cl}(a_1 a_2 x)$.

Make a free amalgam with the following graph over x and embed it into M over X as a closed set:



For each (isomorphic image) z of a leaf of this graph, easy calculation shows that $a_1 a_2 z < M$. Hence, $z \perp a_1 a_2$.

Therefore, $z \in \text{cl}(a_1 a_2, D)$ by Claim 1.

Also $x \in \text{cl}(\{z \mid z \text{ is a leaf above}\})$, thus $x \in \text{cl}(a_1 a_2, D)$.

Main Theorem

We can do similar arguments when $1/2 < \alpha < 2/3$, or $n - 1/n < \alpha < n/n + 1$ for some integer $n \geq 1$.

If $\alpha = n/n + 1$, we can argue similarly, except the case $\alpha = 1/2$.

When $\alpha = 1/2$, the proof by Evans, Ghadernezhad, Tent works (They made some extra assumption on f which fails for Hrushovski's f , but it should work).

Corollary (K.)

Assume that

$0 < \alpha = c/d < 1$ with coprime positive integers c and d ,

$f(0) = 0$, $f(1) = 1$,

$f'_+(x) \leq 1/(d \cdot x)$ where d is the denominator of α , and

$f'_+(x)$ is non-increasing.

Let M be the Fraïssé limit of $(K_f, <_\alpha)$.

Then $\text{Aut}(M)$ is a simple group.

Let f_α be the Hrushovski's boundary function associated to α . With the assumption above, we have $f(x) \leq f_\alpha(x)$ for $x \geq 2$.

Let M_α be the Hrushovski's pseudoplane associated to α .

- $Th(M_\alpha)$ is model complete if α is a rational number with $0 < \alpha < 1$.

Suppose α is an irrational number.

- Is $Aut(M_\alpha)$ a simple group?
- If Hrushovski's boundary function associated to α is bounded, then $Th(M_\alpha)$ is not model complete.
- If Hrushovski's boundary function associated to α is unbounded, is $Th(M_\alpha)$ model complete? (My conjecture is yes.)

1. D. Evans, Z. Ghadernezhad, K. Tent, Simplicity of the automorphism groups of some Hrushovski constructions, *Annals of Pure and Applied Logic* **167** (2016), 22–48.
2. E. Hrushovski, A stable \aleph_0 -categorical pseudoplane, preprint, 1988.
3. H. Kikyo, Model Completeness of Generic Graphs in Rational Cases, *Archive for Mathematical Logic* **57** (7-8), 769–794 (2017).
4. H. Kikyo, S. Okabe, On Automorphism Groups of Hrushovski's Pseudoplanes in Rational Cases, in preparation.
5. F.O. Wagner, Relational structures and dimensions, In *Automorphisms of first-order structures*, Clarendon Press, Oxford, 153–181 (1994).
6. F.O. Wagner, *Simple Theories*, Kluwer, 2000.