Pseudocompact unitary representations of finitely generated groups

Aleksander Ivanov

Institute of Mathematics
Silesian University of Technology

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Hilbert spaces over $\mathbb{R}$

We identify a **Hilbert space** over $\mathbb{R}$ with a many-sorted metric structure

$$\left( \{B_n\}_{n \in \omega}, 0, \{I_{mn}\}_{m < n}, \{\lambda_r\}_{r \in \mathbb{R}}, +, -, \langle \rangle \right),$$

where

- $B_n$ is the ball of elements of norm $\leq n$,
- $I_{mn} : B_m \to B_n$ is the inclusion map,
- $\lambda_r : B_m \to B_{km}$ is scalar multiplication by $r$, with $k$ the unique integer satisfying $k \geq 1$ and $k - 1 \leq |r| < k$;
- $+, - : B_n \times B_n \to B_{2n}$ are vector addition and subtraction and
- $\langle \rangle : B_n \times B_n \to [-n^2, n^2]$ is the binary predicate of the inner product.

The metric on each sort is given by $d(x, y) = \sqrt{\langle x - y, x - y \rangle}$. Every operation uniformly continuous; the continuity moduli are standard.
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Hilbert spaces over $\mathbb{C}$

This approach can be extended to complex Hilbert spaces.

$\left( \{B_n\}_{n \in \omega}, 0, \{l_{mn}\}_{m < n}, \{\lambda_c\}_{c \in \mathbb{C}}, +, -, \langle \rangle Re, \langle \rangle Im \right)$,

- We only extend the family $\lambda_r : B_m \rightarrow B_{km}, r \in \mathbb{R}$, to a family $\lambda_c : B_m \rightarrow B_{km}, c \in \mathbb{C}$, of scalar products by $c \in \mathbb{C}$, with $k$ the unique integer satisfying $k \geq 1$ and $k - 1 \leq |c| < k$.
- The inner product is represented by two predicates: Re- and Im-parts of the inner product.

Infinite dimensional Hilbert spaces are axiomatizable as follows:

$$\inf_{x_1, \ldots, x_n} \max_{1 \leq i < j \leq n}(|\langle x_i, x_j \rangle - \delta_{i,j}|) = 0,$$

$$\delta_{i,j} \in \{0, 1\} \text{ with } \delta_{i,j} = 1 \iff i = j,$$

It is known that this class is $\kappa$-categorical for all infinite $\kappa$, and the corresponding continuous theory admits elimination of quantifiers.
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This approach can be extended to complex Hilbert spaces.

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It is known that this class is $\kappa$-categorical for all infinite $\kappa$, and the corresponding continuous theory admits elimination of quantifiers.
To study unitary representations of finitely generated groups we fix a natural number $t$ and consider the class of *dynamical Hilbert spaces* in the extended signature

$$\left( \{ B_n \}_{n \in \omega}, 0, \{ I_{mn} \}_{m < n}, \{ \lambda_c \}_{c \in \mathbb{Q}[i]}, +, -, \langle \rangle_{Re}, \langle \rangle_{Im}, U_1, \ldots, U_t \right),$$

where $U_j$, $1 \leq j \leq t$, are symbols of unitary operators of $\mathbb{H}$.

- We may assume that all $U_j$ are defined only on $B_1$.
- We add to each $U_i$ the symbol $U'_i$ for the operator $U_i^{-1}$.
- Then we also add the axioms

$$\sup_{v \in B_1} d(U'_i U_i(v), v) \leq 0 \text{ and } \sup_{v \in B_1} d(U_i U'_i(v), v) \leq 0.$$
To study unitary representations of finitely generated groups we fix a natural number $t$ and consider the class of dynamical Hilbert spaces in the extended signature

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Unitary representations

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Pseudocompactness. Problem.

Is every unitary representation of a $t$-generated group pseudocompact as a structure of the form

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i.e. is it elementarily equivalent to a metric ultraproduct of structures of this form which correspond to finite dimensional representations?
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The **metric** in the ultraproduct $\prod_I (X_i, d_i)/D$ is defined by

$$d(((x_i)_I, (x'_i)_I) = \lim_{i \to D} d_i(x_i, x'_i),$$

i.e. by the rule that the distance between $(x_i)_I$ and $(x'_i)_I$ is in the interval $(\varepsilon_1, \varepsilon_2)$ if and only if the set $\{i : d_i(x_i, x'_i) \in (\varepsilon_1, \varepsilon_2)\}$ belongs to the ultrafilter $D$.

$\prod_I (X_i, d_i)/D$ consists of classes of the relation $d(((x_i)_I, (y_i)_I) = 0$. 
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Ultraproducts

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Let $\mathcal{K}$ be a class of metric groups with metrics bounded by some number $s$. We say that a group $G$ is $\mathcal{K}$-approximable if it embeds into a metric ultraproduct of groups from $\mathcal{K}$.

Let $\mathcal{K}$ consist of unitary groups $U(n)$ together with the metric induced by the operator norm $\| T \|_{op} = \sup_{\| v \|=1} \| Tv \|$. We put $d(T, Q) = \| T - Q \|_{op}$.

Groups approximable by these metric groups are called MF (matricial field).

It is an open question if there are non-MF groups.

A. Tikuisis, S. White and W. Winter proved that amenable groups are MF.

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Property MF

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Theorem

Let $G = \langle g_1, \ldots, g_n \rangle$ be a finitely generated group. The group $G$ is MF if and only if there is a dynamical Hilbert space in the signature

$$\left( \{ B_l \}_{l \in \omega}, 0, \{ I_{kl} \}_{k < l}, \{ \lambda_c \}_{c \in \mathbb{Q}[i]}, +, -, \langle \rangle \text{Re}, \langle \rangle \text{Im}, U_1, U_2, \ldots, U_n \right)$$

which is an ultraproduct of finite dimensional dynamical Hilbert spaces of the same signature and the group $\langle U_1, \ldots, U_n \rangle$ is isomorphic to $G$ under the map taking $U_i$ to $g_i$, $1 \leq i \leq n$.

Corollary. If every unitary representation of a finitely generated group is pseudocompact, then every group satisfies property MF.
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Other observations

- Any dynamical Hilbert space corresponding to a representation of a cyclic group is pseudocompact. (Spectral decomposition theorem and previous results of Henson, Argoty and Berenstein.)

- Any unitary representation of a finitely generated group which is existentially closed as a dynamical Hilbert space is pseudocompact.
The **left regular** representation of $G$ is obtained by the action of $G$ on $l^2(G)$ defined by the unitary operators $U_g : f(h) \mapsto f(g^{-1}h)$.

**Theorem**

Let $G$ be a finitely generated LEF group. Then the dynamical $G$-space $l^2(G)$ is pseudo finite dimensional.

- A group $H$ is called LEF if for every finite $F \subseteq H$ there is a finite group $S \supseteq F$ so that $\forall a, b, c \in F \ H \models a \cdot b = c$ if and only if $S \models a \cdot b = c$.
- Residually finite groups are LEF.
- **Corollary.** Any finitely generated LEF group is MF (Carrion, Dadarlat, Eckhardt, 2013).
- This together with some results of Berenstein imply that when $G$ is a f.g. amenable LEF group, then all existentially closed unitary $G$-representations are pseudocompact.
Regular representations

The **left regular** representation of \( G \) is obtained by the action of \( G \) on \( l^2(G) \) defined by the unitary operators \( U_g : f(h) \to f(g^{-1}h) \).

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Universal statements

**Theorem**

*Any dynamical Hilbert space corresponding to a representation of a finitely generated group is embeddable into a metric ultraproduct of finite dimensional unitary representations.*

**Corollary.** Any statement of the form

$$\sup_{x_1} \ldots \sup_{x_n} \phi(\bar{x}) \leq q,$$

where $\phi$ is quantifier free and $q \in \mathbb{Q}$,

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