Pseudocompact unitary representations of finitely generated groups

Aleksander Ivanov

Institute of Mathematics
Silesian Universyty of Technology

August 16, 2019

Hilbert spaces over $\mathbb{R}$

We identify a **Hilbert space** over $\mathbb{R}$ with a many-sorted metric structure

$$(\{B_n\}_{n\in\omega}, 0, \{I_{mn}\}_{m<n}, \{\lambda_r\}_{r\in\mathbb{R}}, +, -, \langle\rangle)\ ,\ \text{where}$$

- B_n is the ball of elements of norm $\leq n$,
- $I_{mn} : B_m \to B_n$ is the inclusion map,
- $\lambda_r : B_m \to B_{km}$ is scalar multiplication by r, with k the unique integer satisfying $k \geq 1$ and $k-1 \leq |r| < k$;
- $+, - : B_n \times B_n \to B_{2n}$ are vector addition and subtraction and
- $\langle\rangle : B_n \times B_n \to [-n^2, n^2]$ is the binary predicate of the inner product.

The metric on each sort is given by $d(x,y) = \sqrt{\langle x-y, x-y\rangle}$. Every operation uniformly continuous; the continuity moduli are standard.

Hilbert spaces over $\mathbb{R}$

We identify a **Hilbert space** over $\mathbb{R}$ with a many-sorted metric structure

$$(\{B_n\}_{n\in\omega}, 0, \{I_{mn}\}_{m<n}, \{\lambda_r\}_{r\in\mathbb{R}}, +, -, \langle\rangle)\ ,\ \text{where}$$

- B_n is the ball of elements of norm $\leq n$,
- $I_{mn} : B_m \to B_n$ is the inclusion map,
- $\lambda_r : B_m \to B_{km}$ is scalar multiplication by r, with k the unique integer satisfying $k \geq 1$ and $k - 1 \leq |r| < k$;
- $+, - : B_n \times B_n \to B_{2n}$ are vector addition and subtraction and
- $\langle\rangle : B_n \times B_n \to [-n^2, n^2]$ is the binary predicate of the inner product.

The metric on each sort is given by $d(x, y) = \sqrt{\langle x - y, x - y\rangle}$.
Every operation uniformly continuous; the continuity moduli are standard.

Hilbert spaces over $\mathbb{R}$

We identify a **Hilbert space** over $\mathbb{R}$ with a many-sorted metric structure

$$(\{B_n\}_{n\in\omega}, 0, \{I_{mn}\}_{m<n}, \{\lambda_r\}_{r\in\mathbb{R}}, +, -, \langle\rangle)\ \text{, where}$$

- B_n is the ball of elements of norm $\leq n$,
- $I_{mn} : B_m \to B_n$ is the inclusion map,
- $\lambda_r : B_m \to B_{km}$ is scalar multiplication by r, with k the unique integer satisfying $k \geq 1$ and $k-1 \leq |r| < k$;
- $+, - : B_n \times B_n \to B_{2n}$ are vector addition and subtraction and
- $\langle\rangle : B_n \times B_n \to [-n^2, n^2]$ is the binary predicate of the inner product.

The metric on each sort is given by $d(x,y) = \sqrt{\langle x-y, x-y\rangle}$. Every operation uniformly continuous; the continuity moduli are standard.

Hilbert spaces over $\mathbb{C}$

This approach can be extended to complex Hilbert spaces.

$$(\{B_n\}_{n\in\omega}, 0, \{I_{mn}\}_{m<n}, \{\lambda_c\}_{c\in\mathbb{C}}, +, -, \langle\rangle_{Re}, \langle\rangle_{Im}),$$

- We only extend the family $\lambda_r : B_m \to B_{km}$, $r \in \mathbb{R}$, to a family $\lambda_c : B_m \to B_{km}$, $c \in \mathbb{C}$, of scalar products by $c \in \mathbb{C}$, with k the unique integer satisfying $k \geq 1$ and $k - 1 \leq |c| < k$.
- The inner product is represented by two predicates: *Re*- and *Im*-parts of the inner product.

Iinfinite dimensional Hilbert spaces are axiomatizable as follows:

$$inf_{x_1,\ldots,x_n} max_{1\leq i<j\leq n}(|\langle x_i, x_j\rangle - \delta_{i,j}|) = 0,$$

$$\delta_{i,j} \in \{0,1\} \text{ with } \delta_{i,j} = 1 \leftrightarrow i = j,$$

It is known that this class is κ-**categorical** for all infinite κ, and the corresponding continuous theory admits elimination of quantifiers.

Hilbert spaces over $\mathbb{C}$

This approach can be extended to complex Hilbert spaces.

$$(\{B_n\}_{n\in\omega}, 0, \{I_{mn}\}_{m<n}, \{\lambda_c\}_{c\in\mathbb{C}}, +, -, \langle\rangle_{Re}, \langle\rangle_{Im}),$$

- We only extend the family $\lambda_r : B_m \to B_{km}$, $r \in \mathbb{R}$, to a family $\lambda_c : B_m \to B_{km}$, $c \in \mathbb{C}$, of scalar products by $c \in \mathbb{C}$, with k the unique integer satisfying $k \geq 1$ and $k - 1 \leq |c| < k$.
- The inner product is represented by two predicates: *Re*- and *Im*-parts of the inner product.

Iinfinite dimensional Hilbert spaces are axiomatizable as follows:

$$inf_{x_1,\ldots,x_n} max_{1\leq i<j\leq n}(|\langle x_i, x_j\rangle - \delta_{i,j}|) = 0,$$

$$\delta_{i,j} \in \{0, 1\} \text{ with } \delta_{i,j} = 1 \leftrightarrow i = j,$$

It is known that this class is κ-**categorical** for all infinite κ, and the corresponding continuous theory admits elimination of quantifiers.

Unitary representations

To study unitary representations of finitely generated groups we fix a natural number t and consider the class of *dynamical Hilbert spaces* in the extended signature

$$(\{B_n\}_{n\in\omega}, 0, \{I_{mn}\}_{m<n}, \{\lambda_c\}_{c\in\mathbb{Q}[i]}, +, -, \langle\rangle_{Re}, \langle\rangle_{Im}, U_1, \ldots, U_t),$$

where U_j, $1 \leq j \leq t$, are symbols of unitary operators of $\mathbb{H}$.

- We may assume that all U_j are defined only on B_1.
- We add to each U_i the symbol U_i' for the operator U_i^{-1}.
- Then we also add the axioms

$$\sup_{v\in B_1} d(U_i'U_i(v), v) \leq 0 \text{ and } \sup_{v\in B_1} d(U_iU_i'(v), v) \leq 0.$$

Unitary representations

To study unitary representations of finitely generated groups we fix a natural number t and consider the class of *dynamical Hilbert spaces* in the extended signature

$$(\{B_n\}_{n\in\omega}, 0, \{I_{mn}\}_{m<n}, \{\lambda_c\}_{c\in\mathbb{Q}[i]}, +, -, \langle\rangle_{Re}, \langle\rangle_{Im}, U_1, \ldots, U_t),$$

where U_j, $1 \leq j \leq t$, are symbols of unitary operators of $\mathbb{H}$.

- We may assume that all U_j are defined only on B_1.
- We add to each U_i the symbol U_i' for the operator U_i^{-1}.
- Then we also add the axioms

$$\sup_{v\in B_1} d(U_i'U_i(v), v) \leq 0 \text{ and } \sup_{v\in B_1} d(U_iU_i'(v), v) \leq 0.$$

Unitary representations

To study unitary representations of finitely generated groups we fix a natural number t and consider the class of *dynamical Hilbert spaces* in the extended signature

$$(\{B_n\}_{n\in\omega}, 0, \{I_{mn}\}_{m<n}, \{\lambda_c\}_{c\in\mathbb{Q}[i]}, +, -, \langle\rangle_{Re}, \langle\rangle_{Im}, U_1, \ldots, U_t),$$

where U_j, $1 \leq j \leq t$, are symbols of unitary operators of $\mathbb{H}$.

- We may assume that all U_j are defined only on B_1.
- We add to each U_i the symbol U_i' for the operator U_i^{-1}.
- Then we also add the axioms

$$\sup_{v\in B_1} d(U_i' U_i(v), v) \leq 0 \text{ and } \sup_{v\in B_1} d(U_i U_i'(v), v) \leq 0.$$

Unitary representations

To study unitary representations of finitely generated groups we fix a natural number t and consider the class of *dynamical Hilbert spaces* in the extended signature

$$(\{B_n\}_{n\in\omega}, 0, \{I_{mn}\}_{m<n}, \{\lambda_c\}_{c\in\mathbb{Q}[i]}, +, -, \langle\rangle_{Re}, \langle\rangle_{Im}, U_1, ..., U_t),$$

where U_j, $1\leq j\leq t$, are symbols of unitary operators of $\mathbb{H}$.

- We may assume that all U_j are defined only on B_1.
- We add to each U_i the symbol U_i' for the operator U_i^{-1}.
- Then we also add the axioms

$$\sup_{v\in B_1} d(U_i'U_i(v), v)\leq 0 \text{ and } \sup_{v\in B_1} d(U_iU_i'(v), v)\leq 0.$$

Pseudocompactness. Problem.

Is every unitary representation of a t-generated group pseudocompact as a structure of the form

$$(\{B_n\}_{n\in\omega}, 0, \{I_{mn}\}_{m<n}, \{\lambda_c\}_{c\in\mathbb{Q}[i]}, +, -, \langle\rangle_{Re}, \langle\rangle_{Im}, U_1, ..., U_t)?$$

i.e. is it elementarily equivalent to a metric ultraproduct of structures of this form which correspond to finite dimensional representations?

Pseudocompactness. Problem.

Is every unitary representation of a t-generated group pseudocompact as a structure of the form

$$(\{B_n\}_{n\in\omega}, 0, \{I_{mn}\}_{m<n}, \{\lambda_c\}_{c\in\mathbb{Q}[i]}, +, -, \langle\rangle_{Re}, \langle\rangle_{Im}, U_1, ..., U_t)?$$

i.e. is it elementarily equivalent to a metric ultraproduct of structures of this form which correspond to finite dimensional representations?

Ultraproducts

The **metric** in the ultraproduct $\prod_I (X_i, d_i)/D$ is defined by

$$d((x_i)_I, (x'_i)_I) = lim_{i \to D} d_i(x_i, x'_i),$$

i.e. by the rule that the distance between $(x_i)_I$ and $(x'_i)_I$ is in the interval $(\varepsilon_1, \varepsilon_2)$ if and only if the set $\{i : d_i(x_i, x'_i) \in (\varepsilon_1, \varepsilon_2)\}$ belongs to the ultrafilter D.

$\prod_I (X_i, d_i)/D$ consists of classes of the relation $d((x_i)_I, (y_i)_I) = 0$.

Ultraproducts

The **metric** in the ultraproduct $\prod_I (X_i, d_i)/D$ is defined by

$$d((x_i)_I, (x'_i)_I) = lim_{i \to D} d_i(x_i, x'_i),$$

i.e. by the rule that the distance between $(x_i)_I$ and $(x'_i)_I$ is in the interval $(\varepsilon_1, \varepsilon_2)$ if and only if the set $\{i : d_i(x_i, x'_i) \in (\varepsilon_1, \varepsilon_2)\}$ belongs to the ultrafilter D.

$\prod_I (X_i, d_i)/D$ consists of classes of the relation $d((x_i)_I, (y_i)_I) = 0$.

Ultraproducts

The **metric** in the ultraproduct $\prod_I (X_i, d_i)/D$ is defined by

$$d((x_i)_I, (x'_i)_I) = lim_{i\to D} d_i(x_i, x'_i),$$

i.e. by the rule that the distance between $(x_i)_I$ and $(x'_i)_I$ is in the interval $(\varepsilon_1, \varepsilon_2)$ if and only if the set $\{i : d_i(x_i, x'_i) \in (\varepsilon_1, \varepsilon_2)\}$ belongs to the ultrafilter D.

$\prod_I (X_i, d_i)/D$ consists of classes of the relation $d((x_i)_I, (y_i)_I) = 0$.

Property MF

- Let $\mathcal{K}$ be a class of metric groups with metrics bounded by some number s. We say that a group G is $\mathcal{K}$-**approximable** if it embeds into a metric ultraproduct of groups from $\mathcal{K}$.
- Let $\mathcal{K}$ consist of unitary groups $U(n)$ together with the metric induced by the operator norm $\parallel T \parallel_{op} = \sup_{\parallel v \parallel = 1} \parallel Tv \parallel$. We put $d(T, Q) = \parallel T - Q \parallel_{op}$.
- Groups approximable by these metric groups are called MF (matricial field).
- It is an open question if there are non-MF groups.
- A. Tikuisis, S. White and W. Winter proved that amenable groups are MF.
- A. Korchagin shows that in many respects property MF is similar to soficity and hyperlinearity.

Property MF

- Let $\mathcal{K}$ be a class of metric groups with metrics bounded by some number s. We say that a group G is $\mathcal{K}$-**approximable** if it embeds into a metric ultraproduct of groups from $\mathcal{K}$.
- Let $\mathcal{K}$ consist of unitary groups $U(n)$ together with the metric induced by the operator norm $\parallel T \parallel_{op} = \sup_{\parallel v \parallel = 1} \parallel Tv \parallel$. We put $d(T, Q) = \parallel T - Q \parallel_{op}$.
- Groups approximable by these metric groups are called MF (matricial field).
- It is an open question if there are non-MF groups.
- A. Tikuisis, S. White and W. Winter proved that amenable groups are MF.
- A. Korchagin shows that in many respects property MF is similar to soficity and hyperlinearity.

Property MF

- Let $\mathcal{K}$ be a class of metric groups with metrics bounded by some number s. We say that a group G is $\mathcal{K}$-**approximable** if it embeds into a metric ultraproduct of groups from $\mathcal{K}$.
- Let $\mathcal{K}$ consist of unitary groups $U(n)$ together with the metric induced by the operator norm $\| T \|_{op} = \sup_{\|v\|=1} \| Tv \|$. We put $d(T, Q) = \| T - Q \|_{op}$.
- Groups approximable by these metric groups are called MF (matricial field).
- It is an open question if there are non-MF groups.
- A. Tikuisis, S. White and W. Winter proved that amenable groups are MF.
- A. Korchagin shows that in many respects property MF is similar to soficity and hyperlinearity.

Property MF

- Let $\mathcal{K}$ be a class of metric groups with metrics bounded by some number s. We say that a group G is $\mathcal{K}$-**approximable** if it embeds into a metric ultraproduct of groups from $\mathcal{K}$.
- Let $\mathcal{K}$ consist of unitary groups $U(n)$ together with the metric induced by the operator norm $\parallel T \parallel_{op} = \sup_{\parallel v \parallel = 1} \parallel Tv \parallel$. We put $d(T, Q) = \parallel T - Q \parallel_{op}$.
- Groups approximable by these metric groups are called MF (matricial field).
- It is an open question if there are non-MF groups.
- A. Tikuisis, S. White and W. Winter proved that amenable groups are MF.
- A. Korchagin shows that in many respects property MF is similar to soficity and hyperlinearity.

Principal observation

Theorem

Let $G = \langle g_1, ..., g_n \rangle$ be a finitely generated group. The group G is MF if and only if there is a dynamical Hilbert space in the signature

$$(\{B_l\}_{l\in\omega}, 0, \{I_{kl}\}_{k<l}, \{\lambda_c\}_{c\in\mathbb{Q}[i]}, +, -, \langle\rangle_{Re}, \langle\rangle_{Im}, U_1, U_2, \ldots, U_n)$$

which is an ultraproduct of finite dimensional dynamical Hilbert spaces of the same signature and the group $\langle U_1, \ldots, U_n \rangle$ is isomorphic to G under the map taking U_i to g_i, $1 \le i \le n$.

Corollary. If every unitary representation of a finitely generated group is pseudocompact, then every group satisfies property MF.

Principal observation

Theorem

Let $G = \langle g_1, ..., g_n \rangle$ be a finitely generated group. The group G is MF if and only if there is a dynamical Hilbert space in the signature

$$(\{B_l\}_{l\in\omega}, 0, \{I_{kl}\}_{k<l}, \{\lambda_c\}_{c\in\mathbb{Q}[i]}, +, -, \langle\rangle_{Re}, \langle\rangle_{Im}, U_1, U_2, \ldots, U_n)$$

which is an ultraproduct of finite dimensional dynamical Hilbert spaces of the same signature and the group $\langle U_1, \ldots, U_n \rangle$ is isomorphic to G under the map taking U_i to g_i, $1 \le i \le n$.

Corollary. If every unitary representation of a finitely generated group is pseudocompact, then every group satisfies property MF.

Other observations

- Any dynamical Hilbert space corresponding to a representation of a cyclic group is pseudocompact. (Spectral decomposition theorem and previous results of Henson, Argoty and Berenstein.)
- Any unitary representation of a finitely generated group which is existentially closed as a dynamical Hilbert space is pseudocompact.

Regular representations

The **left regular** representation of G is obtained by the action of G on $l^2(G)$ defined by the unitary operators $U_g : f(h) \to f(g^{-1}h)$.

Theorem

Let G be a finitely generated LEF group. Then the dynamical G-space $l^2(G)$ is pseudo finite dimensional.

- A group H is called LEF if for every finite $F \subseteq H$ there is a finite group $S \supseteq F$ so that $\forall a, b, c \in F$ $H \models a \cdot b = c$ if and only if $S \models a \cdot b = c$.
- Residually finite groups are LEF.
- **Corollary.** Any finitely generated LEF group is MF (Carrion, Dadarlat, Eckhardt, 2013).
- This together with some results of Berenstein imply that when G is a f.g. amenable LEF group, then all existentially closed unitary G-representations are pseudocompact.

Regular representations

The **left regular** representation of G is obtained by the action of G on $l^2(G)$ defined by the unitary operators $U_g : f(h) \to f(g^{-1}h)$.

Theorem

Let G be a finitely generated LEF group. Then the dynamical G-space $l^2(G)$ is pseudo finite dimensional.

- A group H is called LEF if for every finite $F \subseteq H$ there is a finite group $S \supseteq F$ so that $\forall a, b, c \in F$ $H \models a \cdot b = c$ if and only if $S \models a \cdot b = c$.
- Residually finite groups are LEF.
- **Corollary.** Any finitely generated LEF group is MF (Carrion, Dadarlat, Eckhardt, 2013).
- This together with some results of Berenstein imply that when G is a f.g. amenable LEF group, then all existentially closed unitary G-representations are pseudocompact.

Regular representations

The **left regular** representation of G is obtained by the action of G on $l^2(G)$ defined by the unitary operators $U_g : f(h) \to f(g^{-1}h)$.

Theorem

Let G be a finitely generated LEF group. Then the dynamical G-space $l^2(G)$ is pseudo finite dimensional.

- A group H is called LEF if for every finite $F \subseteq H$ there is a finite group $S \supseteq F$ so that $\forall a, b, c \in F$ $H \models a \cdot b = c$ if and only if $S \models a \cdot b = c$.
- Residually finite groups are LEF.
- **Corollary.** Any finitely generated LEF group is MF (Carrion, Dadarlat, Eckhardt, 2013).
- This together with some results of Berenstein imply that when G is a f.g. amenable LEF group, then all existentially closed unitary G-representations are pseudocompact.

Regular representations

The **left regular** representation of G is obtained by the action of G on $l^2(G)$ defined by the unitary operators $U_g : f(h) \to f(g^{-1}h)$.

Theorem

Let G be a finitely generated LEF group. Then the dynamical G-space $l^2(G)$ is pseudo finite dimensional.

- A group H is called LEF if for every finite $F \subseteq H$ there is a finite group $S \supseteq F$ so that $\forall a, b, c \in F$ $H \models a \cdot b = c$ if and only if $S \models a \cdot b = c$.
- Residually finite groups are LEF.
- **Corollary.** Any finitely generated LEF group is MF (Carrion, Dadarlat, Eckhardt, 2013).
- This together with some results of Berenstein imply that when G is a f.g. amenable LEF group, then all existentially closed unitary G-representations are pseudocompact.

Regular representations

The **left regular** representation of G is obtained by the action of G on $l^2(G)$ defined by the unitary operators $U_g : f(h) \to f(g^{-1}h)$.

Theorem

Let G be a finitely generated LEF group. Then the dynamical G-space $l^2(G)$ is pseudo finite dimensional.

- A group H is called LEF if for every finite $F \subseteq H$ there is a finite group $S \supseteq F$ so that $\forall a, b, c \in F$ $H \models a \cdot b = c$ if and only if $S \models a \cdot b = c$.
- Residually finite groups are LEF.
- **Corollary.** Any finitely generated LEF group is MF (Carrion, Dadarlat, Eckhardt, 2013).
- This together with some results of Berenstein imply that when G is a f.g. amenable LEF group, then all existentially closed unitary G-representations are pseudocompact.

Universal statements

Theorem

Any dynamical Hilbert space corresponding to a representation of a finitely generated group is embeddable into a metric ultraproduct of finite dimensional unitary representations.

Corollary. Any statement of the form

$$\sup_{x_1} \ldots \sup_{x_n} \phi(\bar{x}) \leq q \text{ , where } \phi \text{ is quantifier free and } q \in \mathbb{Q},$$

is satisfied in a representation of a finitely generated group if and only if it is satisfied in a finite dimensional representation of a finitely generated group.

Universal statements

Theorem

Any dynamical Hilbert space corresponding to a representation of a finitely generated group is embeddable into a metric ultraproduct of finite dimensional unitary representations.

Corollary. Any statement of the form

$$\sup_{x_1} \dots \sup_{x_n} \phi(\bar{x}) \leq q \text{ , where } \phi \text{ is quantifier free and } q \in \mathbb{Q},$$

is satisfied in a representation of a finitely generated group if and only if it is satisfied in a finite dimensional representation of a finitely generated group.