Preservation theorems for strong first-order logics

Christian Espíndola

POSTDOCTORAL RESEARCHER AT MASARYK UNIVERSITY - BRNO - CZECH REPUBLIC

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Any universal formula is "preserved under substructures". Conversely, we have:

Theorem

(Łoś-Tarski) If a sentence ϕ of first-order logic is valid in any substructure of a model of ϕ, then it is equivalent to a universal sentence.

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Theorem

(Lyndon) If a sentence ϕ of first-order logic is valid in any homomorphic image of a model of ϕ, then it is equivalent to a positive sentence.

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Strong first-order logics

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$$\forall x_0 \bigwedge_{b_0 \in I} \exists y_0 \bigvee_{c_0 \in I} \forall x_1 \bigwedge_{b_1 \in I} \exists y_1 \bigvee_{c_1 \in I} \qquad \dots \qquad \bigwedge_{i<\omega} \phi_i^{b_0 c_0 b_1 c_1 \dots b_i c_i}(x_0, y_0, \dots, x_i, y_i)$$

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There is a game semantics associated to the sentence (1) as follows: the first player chooses an element x_0 and a conjunct b_0, then the second player chooses an element y_0 and a disjunct c_0, and the game continues with ω many moves, after which the second player wins if with the choices made during the game it is the case that each $\phi_i^{b_0 c_0 b_1 c_1 \ldots b_i c_i}(x_0, y_0, \ldots, x_i, y_i)$ is satisfied in the structure M, for every $i < \omega$.

Since the formula in the matrix corresponds to a closed subset of $|M|^\omega \times I^\omega$, by determinacy for closed games it follows that the game is determined, and hence the formula is said to be true in M if the second player has a winning strategy, while it is said to be false if the first player has a winning strategy, i.e., if:

$$\exists x_0 \bigvee_{b_0 \in I} \forall y_0 \bigwedge_{c_0 \in I} \exists x_1 \bigvee_{b_1 \in I} \forall y_1 \bigwedge_{c_1 \in I} \quad \dots \quad \bigvee_{i<\omega} \neg \phi_i^{b_0 c_0 b_1 c_1 \dots b_i c_i}(x_0, y_0, \dots, x_i, y_i)$$

holds.

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We give here a positive answer to both questions (universal and positive sentences) in the case of Vaught's game logic $\mathcal{L}_{\omega_1,G}$. The methods are however general enough to be carried out within *ZFC* and to apply to a wider variety of the languages presented in the paper.

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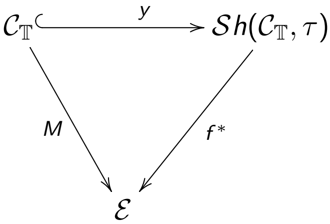
López-Escobar: the theory of well-orderings is not axiomatizable in $\mathcal{L}_{\kappa,\omega}$ for any κ.

The λ-classifying topos of a κ-theory

Every κ-geometric theory has a κ-classifying topos:



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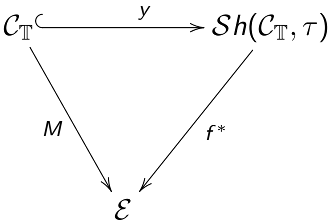

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$$\begin{array}{ccc} \mathcal{C}_{\mathbb{T}} & \xhookrightarrow{\;y\;} & \mathcal{S}h(\mathcal{C}_{\mathbb{T}}, \tau) \\ & {\scriptstyle M}\searrow \qquad \swarrow{\scriptstyle f^*} & \\ & \mathcal{E} & \end{array}$$

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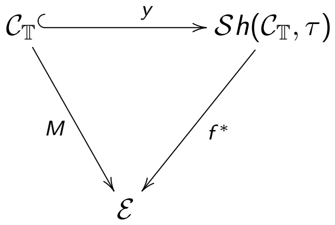

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Theorem

(E.) The λ-classifying topos of $\mathbb{T}'$ is equivalent to the presheaf topos $\mathcal{S}et^{Mod_\lambda(\mathbb{T})}$. Moreover, the canonical embedding of the syntactic category

$$\mathcal{C}_{\mathbb{T}'} \hookrightarrow \mathcal{S}et^{Mod_\lambda(\mathbb{T})}$$

is given by the evaluation functor, which on objects acts by sending $(\mathbf{x}, \phi)$ to the functor $\{M \mapsto [[\phi]]^M\}$.

The λ-classifying topos of a κ-theory

The first consequence is a positive result regarding definability theorems for infinitary logic. If $\mathcal{C}_{\mathcal{T}}$ is the syntactic category of $\mathcal{T}$ considered in $\mathcal{L}_{\lambda^+,\lambda}$, we have that

$$ev : \mathcal{C}_{\mathcal{T}} \longrightarrow \mathcal{S}et^{Mod_{\lambda}(\mathcal{T})}$$

can be identified with Yoneda embedding

$$Y : \mathcal{C}_{\mathcal{T}} \longrightarrow \mathcal{S}h(\mathcal{C}_{\mathcal{T}}, \tau)$$

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Theorem

(Infinitary Beth) Let $\phi(R)$ be a formula in $\mathcal{L}_{\kappa^+,\kappa}$ over the language $\mathcal{L} \cup R$ containing the predicate R. If every $\mathcal{L}$-structure has a unique expansion to a model of $\phi(R)$ and the interpretation of R in each such model is preserved by $\mathcal{L}$-homomorphisms, then there is an $\mathcal{L}$-formula ψ of $\mathcal{L}_{\lambda^+,\lambda}$ such that $R \dashv\vdash_{\mathbf{x}} \psi$.

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- Use an absoluteness argument to generalize the equivalence to all models.
- Use a forcing argument to get rid of the continuum hypothesis.

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Theorem

Let Σ be a signature and consider the category of Σ-structures and homomorphisms. Suppose that for each structure M there is a distinguished subset S^M which is preserved by all homomophisms. Then the following are equivalent:

1. *Vopěnka's principle (every subfunctor of an accessible functor is accessible)*
2. *The subsets S^M are definable by an infinitary coherent formula. That is, there is a formula ϕ of the form $\bigvee_{j<\lambda} \exists_{i<\kappa} x_i \bigwedge_{i<\kappa} \psi_i^j$, where the ψ_i^j are atomic formulas, such that $[[\phi]]^M = S^M$ for all Σ-structures M.*

Thank you!