Proof mining with the bounded functional interpretation

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Outline

Proof mining

Proof mining is a research program that analyzes noneffective proofs in order to obtain new quantitative information using techniques from Proof Theory.

- U. Kohlenbach: monotone functional interpretation (1996)
- General Logic Metatheorems (2003-05)
- F. Ferreira and P. Oliva: bounded functional interpretation (2005)

Metastability

We will look at Cauchy sequences (u_n), i.e.

$$\forall k \in \mathbb{N} \exists n \in \mathbb{N} \forall i, j \geqslant n \left(\|u_i - u_j\| \leqslant \frac{1}{k+1} \right)$$

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In general, it is not possible to guarantee a (computable) bound for n in terms of k.Instead we turn to the "metastable" version,

$$\forall k \in \mathbb{N} \forall f \in \mathbb{N}^{\mathbb{N}} \exists n \in \mathbb{N} \forall i, j \in [n, f(n)] \left(\|u_i - u_j\| \leqslant \frac{1}{k+1} \right),$$

for which we will be able to extract a bound $\phi : \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ such that

$$\forall k \in \mathbb{N} \forall f \in \mathbb{N}^{\mathbb{N}} \exists n \leqslant \phi(k, f) \forall i, j \in [n, f(n)] \left(\|u_i - u_j\| \leqslant \frac{1}{k+1} \right).$$

A theorem by F.E. Browder

Consider X a Hilbert space and a mapping $T : X \to X$. We say that T is **nonexpansive** if $\forall x, y \in X \ (\|T(x) - T(y)\| \leqslant \|x - y\|)$.

Theorem (Browder, 1967)

Let C be a closed, bounded, convex subset of X, u_0 a point in C and $T : C \to C$ a nonexpansive mapping. For each $n \in \mathbb{N}$ consider the strict contraction defined by $T_n(x) := \frac{1}{n+1} u_0 + (1 - \frac{1}{n+1}) T(x)$ and let u_n be its unique fixed point.
Then (u_n) converges strongly to a fixed point of T, the closest to u_0.

The projection argument

We denote $F := Fix(T) := \{x \in C \ : \ T(x) = x\}$.
A central point in Browder's original proof is a projection argument:

$$\exists x \in F \forall y \in F \left(\|u_0 - x\| \leqslant \|u_0 - y\| \right).$$

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Kohlenbach remarked that the following already suffices

$$\forall k \in \mathbb{N} \exists x \in F \forall y \in F \left(\|u_0 - x\| \leqslant \|u_0 - y\| + \frac{1}{k+1} \right).$$

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With $b \geqslant \mathrm{diam}(C)$ and $r := b(k+1)$, we get for all $k \in \mathbb{N}$ and all $f : \mathbb{N} \to \mathbb{N}$ monotone, there are $n \leqslant f^{(r)}(0)$ and $x \in C$ such that

$$\|T(x) - x\| \leqslant \frac{1}{f(n)+1} \text{ and}$$

$$\forall y \in C \left(\|T(y) - y\| \leqslant \frac{1}{n+1} \to \|u_0 - x\| \leqslant \|u_0 - y\| + \frac{1}{k+1} \right).$$

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Weak seq. compactness (and the demiclosedness principle) is used to show:

$$\limsup \langle P_F(u_0) - u_0, P_F(u_0) - u_n \rangle \leqslant 0, \text{ i.e.}$$

$$\forall k \in \mathbb{N} \exists n \in \mathbb{N} \forall m \geqslant n \left(\langle P_F(u_0) - u_0, P_F(u_0) - u_m \rangle \leqslant \frac{1}{k+1} \right).$$

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We don't have access to $P_F(u_0)$. Instead we want to show

$$\forall k \in \mathbb{N} \exists x \in F \exists n \in \mathbb{N} \forall m \geqslant n \left(\langle x - u_0, x - u_m \rangle \leqslant \frac{1}{k+1} \right).$$

This statement can be shown without invoking weak sequential compactness and using instead a collection argument characteristic of the bounded functional interpretation (or **UB** in the context of the monotone functional interpretation).

Theorem (Ferreira-Leuștean-Pinto, 2019)

(On a formal system $\mathcal{T}^{+}_{\mathcal{M}}$) Consider $\varphi : X \times X \to \mathbb{R}$, $T : X \to X$ *and* (u_n) *a sequence of elements of* X *such that* $d(T(u_n), u_n) \to 0$. *If* $\forall k \in \mathbb{N} \exists x \in F \forall y \in F \left(\varphi(x,x) \leqslant \varphi(x,y) + \frac{1}{k+1}\right)$, *then*
$\forall k \in \mathbb{N} \exists x \in F \exists n \in \mathbb{N} \forall m \geqslant n \left(\varphi(x,x) \leqslant \varphi(x,u_m) + \frac{1}{k+1}\right)$.

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$$\forall k \in \mathbb{N} \exists x \in F \exists n \in \mathbb{N} \forall m \geqslant n \left(\varphi(x,x) \leqslant \varphi(x,u_m) + \frac{1}{k+1}\right).$$

E.g., for Browder, $\varphi(x,y) = \langle u_0 - x, y\rangle$.

In the end, the quantitative final version, this collection argument disappears – the idea is similarly to that of Harvey Friedman's conservation result of WKL_0 over RCA_0.

Suppose the existence of monotone functions α and β satisfying:

(a) $\forall k \in \mathbb{N} \tilde{\forall} f : \mathbb{N} \to \mathbb{N} \, \exists N \leqslant \alpha(k, f)$

$$\forall n \in [N, f(N)] \, (d(u_n, T(u_n)) \leqslant \frac{1}{k+1});$$

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(b) $\forall k \in \mathbb{N}\, \tilde{\forall} f : \mathbb{N} \to \mathbb{N}\, \exists N \leqslant \beta(k, f)\, \exists x \in X$
$$\Big(d(x, T(x)) \leqslant \frac{1}{f(N)+1} \wedge \forall y \in X$$
$$(d(y, T(y)) \leqslant \frac{1}{N+1} \to \varphi(x, x) \leqslant \varphi(x, y) + \frac{1}{k+1})\Big).$$

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Then $\forall k \in \mathbb{N}\, \tilde{\forall} f : \mathbb{N} \to \mathbb{N}\, \exists N \leqslant \psi(k, f)\, \exists x \in X$

$$d(x, T(x)) \leqslant \frac{1}{f(N)+1} \wedge \forall n \in [N, f(N)]\, (\varphi(x,x) \leqslant \varphi(x, u_n) + \frac{1}{k+1}),$$

where $\psi(k, f) := \alpha\big(\beta(k, \tilde{f}), f\big)$, with $\tilde{f}(m) := f\big(\alpha(m, f)\big)$

Theorem (Quantitative Browder)

Under the conditions of Browder's theorem, let $b \in \mathbb{N}$ be an upper bound on the diameter of C. Then, for all $k \in \mathbb{N}$ and every monotone function $f : \mathbb{N} \to \mathbb{N}$,

$$\exists N \leqslant \phi_b(k,f) \forall i,j \in [N, f(N)] \left(\|u_i - uj\| \leqslant \frac{1}{k+1} \right),$$

where $\phi_b(k,f) := 12b^2(h^{(R)}(0)+1)^2 + b$,
with $R := 64b^4(k+1)^4$ *and*
$h(m) := \max\{8b(f(12b^2(m+1)^2+b)+1)(k+1)^2-1; 12b(m+1)^2\}$.

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The extracted bound does not depend on the Hilbert space X neither on the map T. The dependency on C is only in the form of a bound b for its diameter.

A theorem by H. Bauschke

Let X be a real Hilbert space and $C \subseteq X$ is a nonempty, closed, convex and bounded subset. Let $T_0, \cdots, T_{\ell-1} : C \to C$ be $\ell \geqslant 1$ nonexpansive mappings.
For any $n \in \mathbb{N}$, define the maps

$$U_n := T_{n \bmod \ell}.$$

and assume

$$\begin{aligned} F := \bigcap_{i \leqslant \ell-1} Fix(U_i) &= Fix(U_{\ell-1} \cdots U_1 U_0) = \\ &= Fix(U_0 U_{\ell-1} \cdots U_1) = \\ &= \cdots = Fix(U_{\ell-2} \cdots U_0 U_{\ell-1}) \neq \emptyset. \end{aligned}$$

Consider a sequence (x_n) defined by:

$$x_0 \in C, \quad x_{n+1} := \lambda_{n+1} x_0 + (1 - \lambda_{n+1}) U_{n+1}(x_n) \text{ with } (\lambda_n) \subset [0, 1].$$

Theorem (Bauschke, 1996)

Under the previous hypothesis, if $(\lambda_n) \subset]0, 1[$ *satisfies:*

1. $\lim_n \lambda_n = 0$;
2. $\sum_n (\lambda_n) = +\infty$;
3. $\sum_n |\lambda_n - \lambda_{n+\ell}| < +\infty$;

then the sequence (x_n) *strongly converges to* $P_F(x_0)$.

This result extends the well-known convergence result by Wittmann $(\ell = 1)$.

Quantitative Bauschke: conditions

For a quantitative version of the condition on the set of common fixed points, we ask for a monotone function $\tau : \mathbb{N} \to \mathbb{N}$ satisfying

$$\forall k \in \mathbb{N}\, \forall m \in \mathbb{N}\, \forall u \in C$$

$$\|u - U_{m+\ell} \cdots U_{m+1}(u)\| \leqslant \frac{1}{\tau(k)+1} \to \forall i < \ell \, \|u - U_i(u)\| \leqslant \frac{1}{k+1}.$$

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For the quantitative version of the conditions on the sequence $(\lambda_n)_n$ we assume the existence of monotone function μ, ν and ξ satisfying:

1. $\forall k \in \mathbb{N}\, \forall n \geqslant \mu(k)\, (\lambda_n \leqslant \frac{1}{k+1})$;
2. $\forall k \in \mathbb{N}\, \Big(\sum_{i=0}^{\nu(k)} \lambda_i \geqslant k \Big)$;
3. $\forall k \in \mathbb{N}\, \forall n \in \mathbb{N}\, \Big(\sum_{i=\xi(k)+1}^{\xi(k)+n} |\lambda_i - \lambda_{i+\ell}| \leqslant \frac{1}{k+1} \Big)$.

The quantitative version of Bauschke's theorem:

Theorem (Quantitative Bauschke, general (λ_n))

Under the previous conditions and with the functions as before we have, for any $k \in \mathbb{N}$ and monotone function $f : \mathbb{N} \to \mathbb{N}$,

$$\exists N \leqslant \phi_{b,\tau,\mu,\nu,\xi}(k, f)\, \forall i, j \in [N, f(N)]\, (\|x_i - x_j\| \leqslant \frac{1}{k+1}).$$

Since for $(\lambda_n) \subset]0, 1[$, we have $\sum \lambda_n = \infty$ is equivalent to $\prod(1 - \lambda_n) = 0$, it was also possible to obtain a similar bound that uses a rate of convergence $\nu' : \mathbb{N} \to \mathbb{N}$ for $(\prod_{i=0}^{n}(1 - \lambda_i))$ towards zero, instead of ν.

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(See also Daniel Körnlein's PhD thesis where he analyzed a generalization of this result due to Yamada.)

Proximal point algorithm

A multi-valued function $A : X \to 2^X$ is said to be **monotone** if

$$\forall x, x' \in X \forall y \in A(x), y' \in A(x'), \langle x - x', y - y' \rangle \geqslant 0.$$

A monotone operator is **maximally monotone** if its graph is not strictly contained in the graph of any monotone operator. Let $zer(A) := \{x \in X \ : \ 0 \in A(x)\}$ denote the set of zeros of A.

One major question: **How to find a zero of A?**

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For each $\beta > 0$, the single-valued resolvent function $J_\beta = (Id + \beta A)^{-1}$ is nonexpansive and

$$Fix(J_\beta) = zer(A).$$

Variants of PPA

$$\text{(PPA)} \qquad x_{n+1} := J_{\beta_n}(x_n)$$

- (Rockafellar) The iteration (PPA) is weakly convergent;
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- Variations of (PPA) to try unsure strong convergence:

$$\text{(HPPA)} \quad x_{n+1} := \lambda_n x_0 + (1-\lambda_n) J_{\beta_n}(x_n)$$

$$\text{(mPPA)} \quad x_{n+1} := \lambda_n u + \gamma_n x_n + \delta_n J_{\beta_n}(x_n)$$

where $(\beta_n) \subset \mathbb{R}^+$, $x_0, u \in X$, $(\lambda_n), (\gamma_n), (\delta_n) \subset]0,1[$ and, in (mPPA), for all $n \in \mathbb{N}$, $\lambda_n + \gamma_n + \delta_n = 1$.

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Projection argument again

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If, instead of C being bounded, we know $F = Fix(J_\beta)$ to be a nonempty set, we can still get a simplified treatment of the projection.

Let $N \geqslant \|u_0 - z\| + \|u_0\|$ for some $z \in F$. Then the original projection argument is equivalent to the one restricted to $F \cap B_N(0)$:

$$\exists x \in F \cap B_N(0)\, \forall y \in F \cap B_N(0)\ (\|u_0 - x\| \leqslant \|u_0 - y\|)$$

HPPA

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(C1) $\lim \lambda_n = 0$;

(C2) $\sum \lambda_n = \infty$;

(C3) $\lim \frac{|\lambda_{n+1}-\lambda_n|}{\lambda_n^2} = 0$;

(C4) $\lim \beta_n = \beta$, for some $\beta > 0$;

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Theorem (Boikanyo-Moroșanu, 2011)

Consider a sequence (x_n) defined by (HPPA) and satisfying (C1)-(C5). Then (x_n) converges strongly to a zero of A.

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Consider a sequence (x_n) defined by (HPPA) and satisfying (C1)-(C5). Then (x_n) converges strongly to a zero of A.

In the proof, the convergence of (x_n) is reduced to that of a sequence (u_n) – an iteration in the "style of Browder".

HPPA: quantitative

(Q1) $\forall k \in \mathbb{N}\, \forall n \geqslant \mu(k)\, (\lambda_n \leqslant \frac{1}{k+1})$;

(Q2) $\forall k \in \mathbb{N} \left(\sum_{i=0}^{\nu(k)} \lambda_i \geqslant k \right)$;

(Q3) $\forall k \in \mathbb{N}\, \forall n \geqslant \xi(k) \left(\frac{|\lambda_{n+1} - \lambda_n|}{\lambda_n^2} \leqslant \frac{1}{k+1} \right)$;

(Q4) $\forall k \in \mathbb{N} \forall n \geqslant B(k)(|\beta_n - \beta| \leqslant \frac{1}{k+1})$;

(Q5) $\forall k, n \in \mathbb{N} \left(\sum_{i=E(k)+1}^{E(k)+n} \|e_i\| \leqslant \frac{1}{k+1} \right)$

We computed a function Θ for:

Theorem (Leuștean-Pinto)

Consider (x_n) defined by (HPPA), β a real number and monotone functions μ, ν, ξ, B and E satisfying respectively (Q1)-(Q5).
Let $b \in \mathbb{N}$ be such that $\beta \geqslant \frac{1}{b+1}$.
Consider (u_n) be the sequence of the fixed points for the strict contractions $T_n(x) := \lambda_n x_0 + (1 - \lambda_n) J_\beta(x)$ and assume (u_n) to be a Cauchy sequence with a bound on its metastable property given by a (monotone) function $\Phi : \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$. Then, for all $k \in \mathbb{N}$ and function $f : \mathbb{N} \to \mathbb{N}$ there is $N \leqslant \Theta[\mu, \nu, \xi, B, E, b, \Phi](k, f)$ s.t.

$$\forall i, j \in [N, f(N)] \left(\|x_i - x_j\| \leqslant \frac{1}{k+1} \right).$$

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In the proof, a certain $\limsup$ plays an essential role.

Limit superior

We want to avoid lim sup:

Lemma

Consider (a_n) be a sequence of real numbers and let $N \in \mathbb{N}$ be such that, for all $n \in \mathbb{N}$, $0 \leqslant a_n \leqslant N$. Then, for all $k \in \mathbb{N}$, there is a natural number $p < N(k+1)$ satisfying

$$\forall n \in \mathbb{N} \exists m \geqslant n \left(x_m \geqslant \frac{p}{k+1} \right) \wedge \exists n' \in \mathbb{N} \forall m' \geqslant n' \left(x_{m'} \leqslant \frac{p+1}{k+1} \right)$$

These rational approximations were enough for the quantitative analysis.

(See also Kohlenbach and Sipoș “The finitary content of sunny nonexpansive retractions”)

mPPA: quantitative

(Q1) $\forall k \in \mathbb{N}\, \forall n \geqslant \mu(k)\, (\lambda_n \leqslant \frac{1}{k+1})$;

(Q2) $\forall k \in \mathbb{N} \left(\sum_{i=0}^{\nu(k)} \lambda_i \geqslant k \right)$;

(Q3) $\forall n \in \mathbb{N}(\frac{1}{a+1} \leqslant \gamma_n \leqslant 1 - \frac{1}{a+1})$;

(Q4) $\forall n \in \mathbb{N}(\beta_n \geqslant \frac{1}{b+1})$;

(Q5) $\forall k \in \mathbb{N} \forall n \geqslant B(k)(|\beta_{n+1} - \beta_n| \leqslant \frac{1}{k+1})$;

(Q6) $\forall k, n \in \mathbb{N} \left(\sum_{i=E(k)+1}^{E(k)+n} \|e_i\| \leqslant \frac{1}{k+1} \right)$

We computed a function Θ for:

Theorem (Dinis-Pinto)

Consider (x_n) defined by (mPPA), a, $b \in \mathbb{N}$ and monotone functions μ, ν, B and E satisfying $(Q1) - (Q6)$. Then, for all $k \in \mathbb{N}$ and function $f : \mathbb{N} \to \mathbb{N}$ there is $N \leqslant \Theta[a, b, \mu, \nu, B, E](k, f)$ s.t.

$$\forall i, j \in [N, f(N)] \left(\|x_i - x_j\| \leqslant \frac{1}{k+1} \right).$$

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Thank you!

Some References

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