Admissibility of Cut for Sequent Calculus related to \( n \)-labelled Tableaux

Andrzej Indrzejczak

Department of Logic, University of Lodz

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VERIFICATIONIST versus FALSIFICATIONIST APPROACH TO SEQUENT CALCULUS

Classical Case:

Falsificationist approach: all elements of $\Gamma$ true and all elements of $\Delta$ false.
Verificationist approach: at least one element of $\Gamma$ false or at least one of $\Delta$ true.

$FA = \Rightarrow$ Hintikka-style tableaux;

$VA = \Rightarrow$ Schütte-style SC (or Rasiowa dual tableaux).
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\[ \Gamma \Rightarrow \Delta \]
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\( FA \implies \) Hintikka-style tableaux;

\( VA \implies \) Schütte-style SC (or Rasiowa dual tableaux).
Some History:

VA: Schröter [1955], Takahashi [1967], Rousseau [1967] ⇒ $\Gamma_1 | \ldots | \Gamma_n$;


We use sequents of the form:

$\Gamma_1 | \ldots | \Gamma_k \Rightarrow \Delta_1 | \ldots | \Delta_n$

where in case of VA:

$\Gamma_1 | \ldots | \Gamma_k$ correspond to $k$ undesignated values;

$\Delta_1 | \ldots | \Delta_n$ to $n$ designated ones;

whereas in FA it is just the opposite.
Some History:

VA: Schröter [1955], Takahashi [1967], Rousseau [1967] →
n-sequents $\Gamma_1 | \ldots | \Gamma_n$;

labelled tableaux.

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\(\Gamma_1 \mid \ldots \mid \Gamma_k\) correspond to \(k\) undesignated values;

\(\Delta_1 \mid \ldots \mid \Delta_n\) to \(n\) designated ones;

whereas in FA it is just the opposite.
Verificationist/Falsificationist Version:

VA: Sequents of the form $\Gamma \mid \Delta \Rightarrow \Sigma$ which are satisfied in a $L_3$-matrix iff some $\phi \in \Gamma$ is 0, or some $\psi \in \Delta$ is 1/2, or some $\chi \in \Sigma$ is 1, under some $h$.

FA: Sequents of the form $\Gamma \Rightarrow \Delta \mid \Sigma$ which are falsified in a $L_3$-matrix iff all $\phi \in \Gamma$ are 1 and all $\psi \in \Delta$ are 1/2, and all $\chi \in \Sigma$ are 0, under some $h$.

VA: Axioms are all sequents with $\Gamma \cap \Delta \cap \Sigma$ nonempty or (in Baaz, et al. version) $\phi \mid \phi \Rightarrow \phi$.

FA: A sequent is counted as an axiom if either $\Gamma \cap \Delta$ or $\Gamma \cap \Sigma$ or $\Delta \cap \Sigma$ is nonempty (or of the form: $\phi \Rightarrow \phi \mid \phi$, $\phi \Rightarrow \mid \phi$, $\Rightarrow \phi \mid \phi$).
Verificationist/Falsificationist Version:

VA: Sequents of the form $\Gamma \mid \Delta \Rightarrow \Sigma$ which are satisfied in a L3-matrix iff some $\varphi \in \Gamma$ is 0, or some $\psi \in \Delta$ is $1/2$, or some $\chi \in \Sigma$ is 1, under some $h$.

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Verificationist/Falsificationist Version – rules:

\((\neg \Rightarrow \Gamma | \Delta \Rightarrow \Sigma, \phi \neg, \Gamma | \Delta \Rightarrow \Sigma (\Rightarrow \neg \Gamma | \phi), \Gamma | \Delta \Rightarrow \Sigma, \neg \phi \neg, \Gamma \Rightarrow \Delta | \Sigma )\)
Verificationist/Falsificationist Version – rules:

\[
\begin{align*}
(\neg | \Rightarrow) & \quad \frac{\Gamma | \Delta \Rightarrow \Sigma, \varphi}{\neg \varphi, \Gamma | \Delta \Rightarrow \Sigma} \\
& \quad \frac{\Gamma | \varphi, \Delta \Rightarrow \Sigma}{\Gamma | \neg \varphi, \Delta \Rightarrow \Sigma} \\
(\Rightarrow \neg) & \quad \frac{\varphi, \Gamma | \Delta \Rightarrow \Sigma}{\Gamma | \Delta \Rightarrow \Sigma, \neg \varphi}
\end{align*}
\]
**Verificationist/Falsificationist Version – rules:**

<table>
<thead>
<tr>
<th>Rule</th>
<th>Premise</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\neg</td>
<td>\Rightarrow$</td>
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\begin{array}{c}
\frac{\Gamma \mid \Delta \Rightarrow \Sigma, \varphi \quad \psi, \Gamma \mid \Delta \Rightarrow \Sigma}{\varphi \rightarrow \psi, \Gamma \mid \Delta \Rightarrow \Sigma} \quad \text{(→⇒)} \\
\frac{\varphi, \Gamma \mid \varphi, \Delta \Rightarrow \Sigma, \psi \quad \varphi, \Gamma \mid \psi, \Delta \Rightarrow \Sigma, \psi}{\Gamma \mid \Delta \Rightarrow \Sigma, \varphi \rightarrow \psi} \quad \text{(⇒→)}
\end{array}
\]

versus

\[
\begin{array}{c}
\frac{\Gamma \mid \varphi, \psi, \Delta \Rightarrow \Sigma \quad \psi, \Gamma \mid \Delta \Rightarrow \Sigma}{\Gamma \mid \varphi \rightarrow \psi, \Delta \Rightarrow \Sigma} \quad \text{(⇒|→)} \\
\frac{\varphi, \Gamma \mid \varphi, \Delta \Rightarrow \Sigma \quad \psi, \Gamma \mid \psi, \Delta \Rightarrow \Sigma}{\Gamma \mid \Delta \Rightarrow \Sigma, \varphi \rightarrow \psi} \quad \text{(|$\Rightarrow$)}
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\frac{\Gamma | \varphi, \psi, \Delta \Rightarrow \Sigma, \psi, \Gamma | \Delta \Rightarrow \Sigma, \varphi}{\Gamma | \varphi \rightarrow \psi, \Delta \Rightarrow \Sigma}
\]

\[
\frac{\varphi, \Gamma | \varphi, \Delta \Rightarrow \Sigma, \psi}{\Gamma | \Delta \Rightarrow \Sigma, \varphi \rightarrow \psi}
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\frac{\varphi, \Sigma \Rightarrow \Delta | \Gamma, \psi}{\Sigma \Rightarrow \Delta | \Gamma, \varphi \rightarrow \psi}
\]

\[
\frac{\Sigma \Rightarrow \Delta, \varphi | \Gamma, \psi}{\varphi, \Sigma \Rightarrow \Delta, \psi | \Gamma}
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\]

versus

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Admissibility of Cut for Sequent Calculus related to $n$-labelled Tableaux
Verificationist/Falsificationist Version – proofs:

VA: $\vdash \phi$ if there is a proof of $\phi$.

FA: $\vdash \phi$ if there is a proof of $\phi$.

Derivability:

FA: $\Gamma \vdash \phi$ if there is a proof of $\Gamma \Rightarrow \phi$.

but!

VA: $\Gamma \vdash \phi$ if there is a proof of $\Pi(\Gamma) \Rightarrow \phi$.

where $\Pi(\Gamma)$ is one of the $\Gamma$ quasi-partitions of $\Gamma$. 
Verificationist/Falsificationist Version – proofs:

Theorems:
Verificationist/Falsificationist Version – proofs:

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Verificationist/Falsificationist Version – proofs:

Theorems:

VA: \( \vdash \varphi \) iff there is a proof of \( \Rightarrow \varphi \)

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Verificationist/Falsificationist Version – proofs:

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but!

VA: $\Gamma \vdash \varphi$ iff there is a proof of $\Pi(\Gamma) \Rightarrow \varphi$

where $\Pi(\Gamma)$ is one of the $2^{|\Gamma|}$ quasi-partitions of $\Gamma$. 
Verificationist/Falsificationist Version – Cut:

\[ \Gamma | \Delta \Rightarrow \Sigma \], \phi, \Gamma | \phi, \Delta \Rightarrow \Sigma \]

versus

\[ \Gamma | \Delta \Rightarrow \Sigma \Rightarrow \Delta | \Gamma, \phi \Rightarrow \Sigma \], \phi | \Gamma \Rightarrow \Sigma \]

For VA cut admissibility proved by Baaz, Fermüller and Zach [1994].

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Veriﬁcationist/Falsiﬁcationist Version – Cut:

\[
\begin{align*}
\Gamma \mid \Delta & \Rightarrow \Sigma, \varphi & \varphi, \Gamma \mid \Delta & \Rightarrow \Sigma \\
\Gamma \mid \Delta & \Rightarrow \Sigma
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\]

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\begin{align*}
\Gamma \mid \Delta & \Rightarrow \Sigma, \varphi & \Gamma \mid \varphi, \Delta & \Rightarrow \Sigma \\
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versus
\[ \frac{\Gamma, \Delta \Rightarrow \Sigma, \varphi \quad \varphi, \Gamma | \Delta \Rightarrow \Sigma}{\Gamma | \Delta \Rightarrow \Sigma} \]

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\[ \frac{\Gamma | \Delta, \varphi \Rightarrow \Sigma \quad \varphi, \Gamma | \Delta \Rightarrow \Sigma}{\Gamma | \Delta \Rightarrow \Sigma} \]

\[ (3 - \text{cut}) \]

\[ \frac{\Sigma \Rightarrow \Delta | \Gamma, \varphi \quad \Sigma \Rightarrow \Delta, \varphi | \Gamma \quad \varphi, \Sigma \Rightarrow \Delta | \Gamma}{\Gamma \Rightarrow \Delta | \Sigma} \]
Veriﬁcationist/Falsiﬁcationist Version – Cut:

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\frac{\Gamma \mid \Delta \Rightarrow \Sigma, \varphi \quad \varphi, \Gamma \mid \Delta \Rightarrow \Sigma}{\Gamma \mid \Delta \Rightarrow \Sigma}
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\[
(3 \text{ - cut}) \quad \frac{\Sigma \Rightarrow \Delta \mid \Gamma, \varphi \quad \Sigma \Rightarrow \Delta, \varphi \mid \Gamma \quad \varphi, \Sigma \Rightarrow \Delta \mid \Gamma}{\Gamma \Rightarrow \Delta \mid \Sigma}
\]

For VA cut admissibility proved by Baaz, Fermüller and Zach [1994].
Cut Admissibility for Falsificationist Version – auxiliary results:

1. Reduction to atomic axioms;
2. H-p admissibility of Weakening (three versions);
3. H-p invertibility of all rules (with respect to all premises);
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Cut Admissibility for Falsificationist Version – auxiliary results:

1. reduction to atomic axioms;
2. h-p admissibility of Weakening (three versions);
3. h-p invertibility of all rules (with respect to all premisses);
4. h-p invertibility of Contraction (in three versions).
Cut Admissibility for Falsificationist Version – reduction to atomic axioms:

\[ \Sigma \Rightarrow \Delta, \phi \mid \Gamma, \psi, \phi \]

\[ \Sigma \Rightarrow \Delta, \phi \mid \Gamma, \psi \]

\[ \Delta \rightarrow \rightarrow \mid \phi \rightarrow \psi, \Sigma \Rightarrow \Delta, \phi \rightarrow \psi \mid \Gamma, \phi \rightarrow \psi. \]
Cut Admissibility for Falsificationist Version – reduction to atomic axioms:

\[ \Sigma \Rightarrow \Delta, \varphi | \Gamma, \psi, \varphi \quad \Sigma \Rightarrow \Delta, \varphi, \varphi, \psi | \Gamma, \psi \quad \psi, \Sigma \Rightarrow \Delta, \varphi | \Gamma, \psi \]

\[ \varphi \rightarrow \psi, \Sigma \Rightarrow \Delta, \varphi | \Gamma, \psi \]

\[ \varphi \rightarrow \psi, \Sigma \Rightarrow \Delta, \varphi \rightarrow \psi | \Gamma \]
Cut Admissibility for Falsificationist Version – reduction to atomic axioms:

\[
\begin{align*}
\frac{\Sigma \Rightarrow \Delta, \varphi | \Gamma, \psi, \varphi}{\varphi 
\to \psi, \Sigma \Rightarrow \Delta, \varphi | \Gamma, \psi} & \quad \frac{\Sigma \Rightarrow \Delta, \varphi, \varphi, \psi | \Gamma, \psi}{\varphi, \Sigma \Rightarrow \Delta, \varphi | \Gamma, \psi} & \quad \frac{\psi, \Sigma \Rightarrow \Delta, \varphi | \Gamma, \psi}{\varphi \to \psi, \varphi, \Sigma \Rightarrow \Delta, \psi} \\
(\Rightarrow \to) & \quad (\Rightarrow \to |) & \quad (\to \Rightarrow)
\end{align*}
\]

the same must be done for \( \varphi \to \psi, \Sigma \Rightarrow \Delta | \varphi \to \psi, \Gamma \) and for \( \Sigma \Rightarrow \Delta, \varphi \to \psi | \Gamma, \varphi \to \psi. \)
Cut Admissibility for Falsificationist Version:

1. at least one premiss axiomatic \( \Rightarrow \) trivial reduction;
2. at least one premiss with parametric cut-formula \( \Rightarrow \) by induction on the height (all rules are substitutive);
3. all cut-formulae principal \( \Rightarrow \) by induction on the complexity of cut-formula (all rules are reductive).
Cut Admissibility for Falsificationist Version:

Dragalin-style proof – three main cases:

1. at least one premiss axiomatic $\Rightarrow$ trivial reduction;
2. at least one premiss with parametric cut-formula $\Rightarrow$ by induction on the height (all rules are substitutive);
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Cut Admissibility for Falsificationist Version – principal cut-formulae:
Cut Admissibility for Falsificationist Version – principal cut-formulae:

\[
\Sigma \Rightarrow \Delta, \varphi \land \psi | \Gamma \quad (\Rightarrow \land | \ \ )
\]

\[
\frac{\Pi \Rightarrow \Lambda | \Theta, \varphi \quad \Pi \Rightarrow \Lambda | \Theta, \psi}{\Pi \Rightarrow \Lambda | \Theta, \varphi \land \psi}
\]

\[
\frac{\varphi, \psi, \Xi \Rightarrow \Upsilon | \Omega}{\varphi \land \psi, \Xi \Rightarrow \Upsilon | \Omega}
\]

\[
\frac{\Sigma, \Pi, \Xi \Rightarrow \Delta, \Lambda, \Upsilon | \Gamma, \Theta, \Omega}{(3 \ - \ Cut)}
\]

where the leftmost premiss is deduced by:

\[
(\Rightarrow \land |)
\]

\[
\frac{\varphi, \Sigma \Rightarrow \Delta, \psi | \Gamma}{\Sigma \Rightarrow \Delta, \varphi, \psi | \Gamma}
\]

\[
\frac{\Sigma \Rightarrow \Delta, \varphi, \psi | \Gamma}{\Sigma \Rightarrow \Delta, \varphi \land \psi | \Gamma}
\]

\[
\frac{\psi, \Sigma \Rightarrow \Delta, \varphi | \Gamma}{(\Rightarrow \land |)}
\]

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Cut Admissibility for Falsificationist Version – principal cut-formulae:

\[ P \Rightarrow \Lambda | \Theta, \psi, \Xi \Rightarrow \Upsilon | \Omega, \phi, \Sigma \Rightarrow \Delta, \psi | \Gamma \]

where the leftmost premiss \( P \) is:

\[ \Pi \Rightarrow \Lambda | \Theta, \psi, \Sigma \Rightarrow \Delta, \phi | \Gamma \]

\[ \Sigma, \Pi, \Xi \Rightarrow \Delta, \Lambda, \Upsilon | \Gamma, \Theta, \Omega \]
Cut Admissibility for Falsificationist Version – principal cut-formulae:

\[
\begin{align*}
& (3 - \text{Cut}) \quad \Sigma, \Sigma, \Pi, \Pi, \Xi \Rightarrow \Delta, \Delta, \Lambda, \Lambda, \Gamma | \Gamma, \Gamma, \Theta, \Omega \\
& (3 - \text{Cut}) \quad \Sigma, \Pi, \Xi \Rightarrow \Delta, \Lambda, \Gamma | \Gamma, \Theta, \Omega
\end{align*}
\]

where the leftmost premiss \( P \) is:

\[
\begin{align*}
& (3 - \text{Cut}) \quad \Pi \Rightarrow \Lambda | \Theta, \psi \\
& \psi, \Sigma \Rightarrow \Delta, \psi | \Gamma \\
& \Sigma \Rightarrow \Delta, \Lambda, \psi | \Gamma
\end{align*}
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