# An embedding of **IPC** into $\mathbf{F_{at}}$ not relying on instantiation overflow

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# System $\mathbf{F_{at}}$: atomic polymorphism $(\wedge, \rightarrow, \forall)$

$$\dfrac{\begin{array}{c}\vdots\\ A\end{array}}{\forall X.A}\ \forall I$$

$X$ does not occur free in any undischarged hypothesis;

$$\dfrac{\begin{array}{c}\vdots\\ \forall X.A\end{array}}{A[\color{red}{Y}/X]}\ \forall E$$

$\color{red}{Y}$ atomic

# Why embed **IPC** into $\mathbf{F_{at}}$?

| **IPC** | $\mathbf{F_{at}}$ | |
|---|---|---|
| $\bot, \wedge, \vee, \to$ | $\wedge, \to, \forall$ | *no bad connectives* |
| | | *no commuting conversions* |
| | | *predicative* |
| | | *strong normalization property* |
| | | *subformula property* |
| | | *disjunction property* |

"The elimination rules [$\bot$, $\vee$] are very bad. What is catastrophic about them is the parasitic presence of a formula $F$ which has no structural link with the formula which is eliminated."

— J.-Y. Girard, *Proofs and Types*, 1989, pages 73-74

# Embedding of **IPC** into $\mathbf{F_{at}}$

$$\begin{array}{ccc} & (\cdot)^* & \\ \mathbf{IPC} & \hookrightarrow & \mathbf{F_{at}} \\ \bot, \wedge, \vee, \to & & \wedge, \to, \forall \end{array}$$

Russell-Prawitz's translation:

$$X^* :\equiv X$$

$$\bot^* :\equiv \forall X.X$$

$$(A \vee B)^* :\equiv \forall X.((A^* \to X) \wedge (B^* \to X)) \to X$$

$$(A \wedge B)^* :\equiv A^* \wedge B^*$$

$$(A \to B)^* :\equiv A^* \to B^*.$$

# Embedding of **IPC** into $\mathbf{F_{at}}$

| **IPC** | $\hookrightarrow$ | $\mathbf{F_{at}}$ |
|---|---|---|
| $\bot, \wedge, \vee, \rightarrow$ | | $\wedge, \rightarrow, \forall$ |

Russell-Prawitz's translation + *instantiation overflow*:

For formulas of the form

$$\forall X.X$$

$$\forall X.((A \rightarrow X) \wedge (B \rightarrow X)) \rightarrow X$$

it is possible to deduce in $\mathbf{F_{at}}$ (respectively)

$$F$$

$$((A \rightarrow F) \wedge (B \rightarrow F)) \rightarrow F,$$

for any (not necessarily atomic) formula $F$.

# IPC $\hookrightarrow$ $\mathbf{F_{at}}$

$$\frac{\begin{array}{c}\vdots\\ A\vee B\end{array}\qquad \begin{array}{c}[A]\\ \vdots\\ F\end{array}\qquad \begin{array}{c}[B]\\ \vdots\\ F\end{array}}{F}\ \vee\mathrm{E}$$

In system $\mathbf{F_{at}}$:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \color{red}{(A\vee B)^* :\equiv}\ \forall X.((A^*\to X)\wedge(B^*\to X))\to X\end{array}}{((A^*\to F)\wedge(B^*\to F))\to F}\qquad \dfrac{\dfrac{\begin{array}{c}[A^*]\\ \vdots\\ F\end{array}}{A^*\to F}\qquad \dfrac{\begin{array}{c}[B^*]\\ \vdots\\ F\end{array}}{B^*\to F}}{(A^*\to F)\wedge(B^*\to F)}}{F}$$

$$\frac{\forall X.((A \to X) \wedge (B \to X)) \to X}{((A \to F) \wedge (B \to F)) \to F}$$

For $F :\equiv C_1 \to C_2$

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall X.((A \to X) \wedge (B \to X)) \to X}{((A \to C_2) \wedge (B \to C_2)) \to C_2}\,\text{I.H.} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[(A \to (C_1 \to C_2)) \wedge (B \to (C_1 \to C_2))]}{A \to (C_1 \to C_2)} \qquad [A]}{C_1 \to C_2} \qquad [C_1]}{C_2}}{A \to C_2} \qquad \dfrac{\mathcal{D}}{B \to C_2}}{(A \to C_2) \wedge (B \to C_2)}}{C_2}}{C_1 \to C_2}}{((A \to (C_1 \to C_2)) \wedge (B \to (C_1 \to C_2))) \to (C_1 \to C_2)}$$

where $\mathcal{D}$ is the deduction

$$\dfrac{\dfrac{\dfrac{[(A \to (C_1 \to C_2)) \wedge (B \to (C_1 \to C_2))]}{B \to (C_1 \to C_2)} \qquad [B]}{C_1 \to C_2} \qquad [C_1]}{C_2}$$

# **IPC** in $\lambda$-calculus notation

The types/formulas are given by

$$A, B, C \ ::= \ X \mid \bot \mid A \to B \mid A \wedge B \mid A \vee B$$

The terms/proofs $M, N, P, Q$ are inductively generated as follows:

$$\begin{array}{lcll} M & ::= & x & \text{(assumption)} \\ & \mid & \lambda x^A.M \mid MN & \text{(implication)} \\ & \mid & \langle M, N \rangle \mid M1 \mid M2 & \text{(conjunction)} \\ & \mid & in_1(M, A, B) \mid in_2(N, A, B) \mid case(M, x^A.P, y^B.Q, C) & \text{(disjunction)} \\ & \mid & abort(M, A) & \text{(absurdity)} \end{array}$$

# $\mathbf{F_{at}}$ in $\lambda$-calculus notation

The types/formulas are given by

$$A, B \ ::= \ X \,|\, A \to B \,|\, A \wedge B \,|\, \forall X.A$$

The proof terms $M, N$ are inductively generated as follows:

$$\begin{array}{lcll} M & ::= & x & \text{(assumption)} \\ & | & \lambda x^A.M \,|\, MN & \text{(implication)} \\ & | & \langle M, N\rangle \,|\, M1 \,|\, M2 & \text{(conjunction)} \\ & | & \Lambda X.M \,|\, MX & \text{(universal quantification)} \end{array}$$

# Inference/typing rule of $\mathbf{F_{at}}$

$$\frac{\Gamma \vdash M : \forall X.A}{\Gamma \vdash MY : A[Y/X]} \,\forall E_{at}$$

# Embedding $(\cdot)^*$ of **IPC** into $\mathbf{F_{at}}$ in $\lambda$-calculus notation

Given $M \in$ **IPC**, $M^*$ is defined by recursion on M:

$$\begin{aligned}
x^* &= x \\
(\lambda x^A.M)^* &= \lambda x^{A^*}.M^* \\
(MN)^* &= M^* N^* \\
\langle M, N\rangle^* &= \langle M^*, N^*\rangle \\
\ldots &= \ldots \\
(case(M, x^A.P, y^B.Q, C))^* &= \underline{io}(M^*, A^*, B^*, C^*)\langle \lambda x^{A^*}.P^*, \lambda y^{B^*}.Q^*\rangle \\
(abort(M, A))^* &= \underline{abort}(M^*, A^*)
\end{aligned}$$

$$\begin{aligned}
\underline{io}(M, A, B, X) &= MX \\
\underline{io}(M, A, B, C_1 \wedge C_2) &= \lambda z.\langle \underline{io}(M, A, B, C_i)\langle \lambda x^A.z1xi, \lambda y^B.z2yi\rangle\rangle_{i=1,2} \\
\underline{io}(M, A, B, C_1 \to C_2) &= \lambda z.\lambda u^{C_1}.\underline{io}(M, A, B, C_2)\langle \lambda x^A.z1xu, \lambda y^B.z2yu\rangle \\
\underline{io}(M, A, B, \forall X.C_1) &= \lambda z.\Lambda X.\underline{io}(M, A, B, C_1)\langle \lambda x^A.z1xX, \lambda y^B.z2yX\rangle
\end{aligned}$$

$$\begin{aligned}
\underline{abort}(M, X) &= MX \\
\underline{abort}(M, A_1 \wedge A_2) &= \langle \underline{abort}(M, A_1), \underline{abort}(M, A_2)\rangle \\
\underline{abort}(M, B \to C) &= \lambda z^B.\underline{abort}(M, C) \\
\underline{abort}(M, \forall X.A) &= \Lambda X.\underline{abort}(M, A)
\end{aligned}$$

# Alternative embedding $(\cdot)^\circ$ of **IPC** into $\mathbf{F_{at}}$

Given $M \in \mathbf{IPC}$, $M^\circ$ is defined by recursion on M:

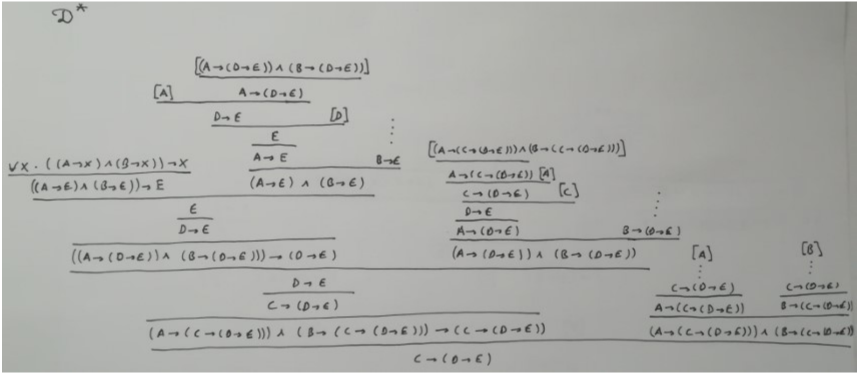
$$\begin{array}{rcl}
x^\circ & = & x \\
(\lambda x^A.M)^\circ & = & \lambda x^{A^\circ}.M^\circ \\
(MN)^\circ & = & M^\circ N^\circ \\
\langle M, N\rangle^\circ & = & \langle M^\circ, N^\circ\rangle \\
\ldots & = & \ldots \\
(case(M, x^A.P, y^B.Q, C))^\circ & = & \underline{case}(M^\circ, x^{A^\circ}.P^\circ, y^{B^\circ}.Q^\circ, C^\circ) \\
(abort(M, A))^\circ & = & \underline{abort}(M^\circ, A^\circ)
\end{array}$$

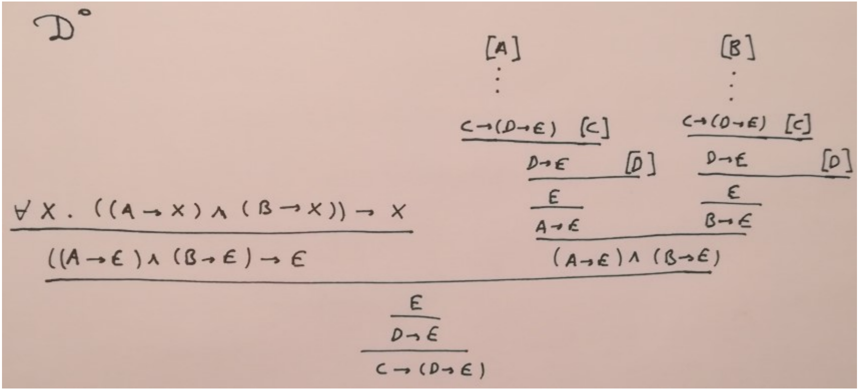
$$\begin{array}{rcl}
\underline{case}(M, x^A.P, y^B.Q, X) & = & MX\langle \lambda x^A.P, \lambda y^B.Q\rangle \\
\underline{case}(M, x^A.P, y^B.Q, C_1 \wedge C_2) & = & \langle \underline{case}(M, x^A.Pi, y^B.Qi, C_i)\rangle_{i=1,2} \\
\underline{case}(M, x^A.P, y^B.Q, C \to D) & = & \lambda z^C.\underline{case}(M, x^A.Pz, y^B.Qz, D) \\
\underline{case}(M, x^A.P, y^B.Q, \forall X.C) & = & \Lambda X.\underline{case}(M, x^A.PX, y^B.QX, C)
\end{array}$$

$$\Gamma \vdash_{\mathbf{IPC}} M : A \quad \Rightarrow \quad \Gamma^\circ \vdash_{\mathbf{F_{at}}} M^\circ : A^\circ$$

Let $\mathcal{D}$ be the following derivation in **IPC**:

$$\dfrac{A \vee B \qquad \begin{matrix}[A] \\ \vdots \\ C \to (D \to E)\end{matrix} \qquad \begin{matrix}[B] \\ \vdots \\ C \to (D \to E)\end{matrix}}{C \to (D \to E)}$$


D*
[(A→(D→E)) ∧ (B→(D→E))]
[A]
A→(D→E)
D→E
[D]
E
A→E
B→E
∀X.((A→X)∧(B→X))→X
((A→E)∧(B→E))→E
(A→E) ∧ (B→E)
E
D→E
((A→(D→E)) ∧ (B→(D→E))) → (D→E)
[(A→(C→(D→E))) ∧ (B→(C→(D→E)))]
A→(C→(D→E))
C→(D→E)
[C]
D→E
A→(D→E)
B→(D→E)
(A→(D→E)) ∧ (B→(D→E))
[B]
D→E
C→(D→E)
C→(D→E)
A→(C→(D→E))
B→(C→(D→E))
(A→(C→(D→E))) ∧ (B→(C→(D→E))) → (C→(D→E))
(A→(C→(D→E))) ∧ (B→(C→(D→E)))
C→(D→E)


D°
[A]
C→(D→E) [C]
D→E [D]
E
A→E
[B]
C→(D→E) [C]
D→E [D]
E
B→E
∀X. ((A→X) ∧ (B→X)) → X
((A→E) ∧ (B→E)) → E
(A→E) ∧ (B→E)
E
D→E
C→(D→E)

# Comparison between $(\cdot)^*$ and $(\cdot)^\circ$

- $(\cdot)^*$ translation of proofs based on instantiation overflow
- $(\cdot)^\circ$ translation of proofs based on the admissibility of the elimination rules for disjunction and absurdity
- both work equally well at the levels of provability and preservation of proof identity
- $(\cdot)^\circ$ produces shorter derivations and shorter simulations of reduction sequences

$$M^* \to_\beta M^\circ$$

J. Espírito Santo, G. Ferreira, *A refined interpretation of intuitionistic logic by means of atomic polymorphism*, **Studia Logica** (2019)

THANK YOU