Logic of Computing with Continuous Data

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Computer Game *Civilization*

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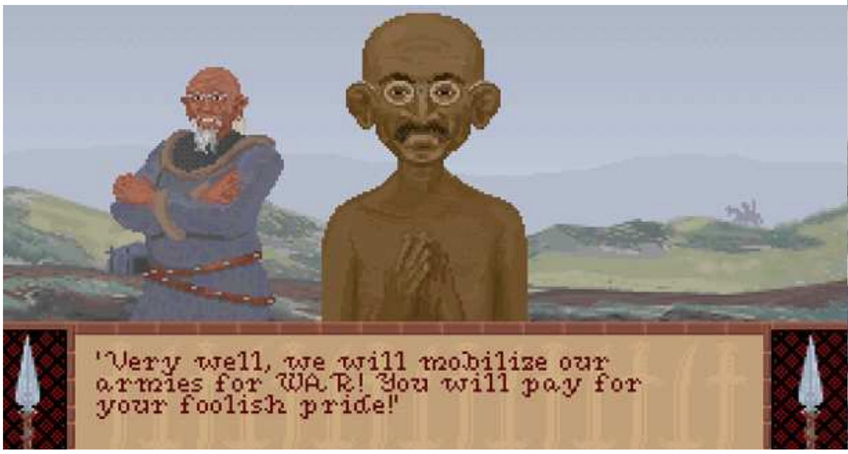
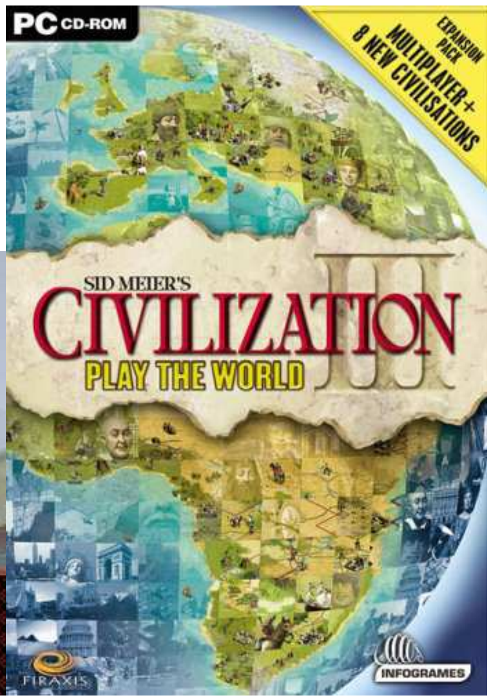
In the simulation, each country (leader) is described by quantitative attributes that govern its "AI" behavior.

Initially Gandhi's **`aggression`** :=1

When a country adopts democracy, its **`aggression`** decreases by 2.

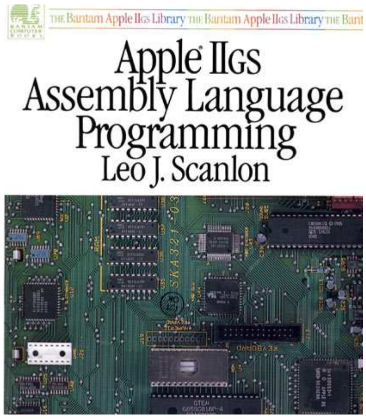




Computer Nostalgia (1980ies)

Martin Ziegler

- *empirical* correctness
- *hardware* data type **`byte/word`** $\neq \mathbb{Z}$
- *In*convenient/ *un*elegant →bugs
- *absolute* performance
- *low*-level prog.language



1 MHz CPU
64kB RAM

Coding, **not** Computer <u>Science</u>

Contemporary Computer <u>Science</u>

Martin Ziegler

- *empirical* correctness
- *hardware* data type **`byte/word`** $\neq \mathbb{Z}$
- *in*convenient/ *un*elegant →bugs
- *absolute* performance
- *low*-level prog.language

1. <u>Rigorous specification</u>
2. Algorithm design&analysis (*always* correct & efficient)
3. <u>Proof of optimality (polynom.-time *reduction*)</u>
4. Design/build on <u>axiomatic</u> *Abstract Data Types*
5. in obj.-oriented high-level prog.language (<u>semantics</u>)
6. <u>Formal verification</u>
7. Implementation/coding

L O G I C

Abstract Data Type = Structure

Martin Ziegler

- *empirical* correctness
- *hardware* data type `byte/word`
- convenient, elegant → *less* bugs
- ~~*absolute* performance~~
- ~~*low*-level prog.language~~

- $\mathbb{Z}$ (`integer` in Python3; `bigint` in Java; GMP)
- `stack` of X
- `queue` of X
- `file` of X
- `array`$[X]$ of Y
- (labeled) graph (of X,Y)

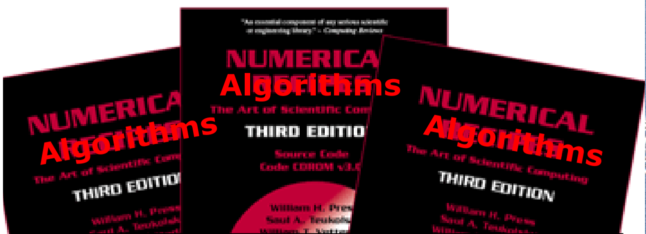
Object-oriented *high*-level programming language:

- *modular* software design
- "*Information hiding*"
- *ideal* data types

Contemporary ***Numerics***

Martin Ziegler

- *empirical* correctness
- *hardware* data type `float/double` ≠$\mathbb{R}$
- *in*convenient/ *un*elegant →bugs
- *absolute* performance



„**nag_opt_one_var_deriv**" *normally* computes a sequence of x values which *tend in the limit* to a minimum of F x

7. Implementation/coding

Medieval Mathematics

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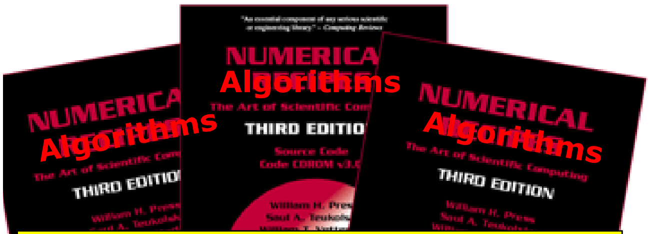
- *empirical* correctness
- *hardware* data type `float/double` ≠ℝ
- *in*convenient/ *un*elegant & ***in***consistent
- *absolute* performance



„nag_opt_one_var_deriv"
normally computes a sequence of x values which *tend in the limit* to a minimum of F x

ℚ, as opposed to ℝ, violates:

- intermed. value theorem
- compactness of [0,1]
- fixed point theorems,
- logical completeness,
- and even distributive law

Don't test for equality!

How about *in*equality "**<**" ?

$x=0 \;\Leftrightarrow\; \neg(x<0) \;\wedge\; \neg(x>0)$

IEEE754:
±∞, ±0, **NaN**s, *de*normals

Example Prog. Codes

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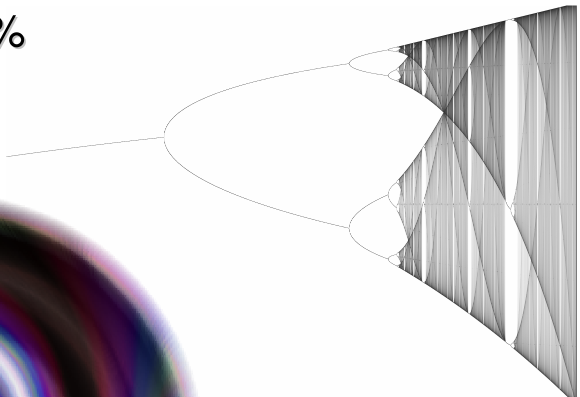
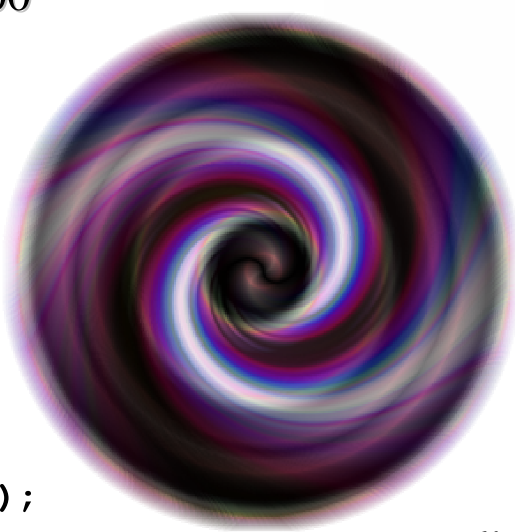
$x_n \to x_{n+1} := r\cdot x_n\cdot(1\text{-}x_n)$ error ≤10%



`float`: $n\leq 30$, `double`: $n\leq 85$,
`long double`: $n\leq 200$
`rational`: $n\leq 24$



```
MATLAB function
  y=jmmuller(m)
x = vpa(11/2);
y = vpa(61/11);
while m>1;
 t = vpa(111 -
    (1130-3000/x)/y);
 x=y; y=t; m=m-1;
 end; end
```
→ **100**

$x_0 := 11/2, \quad x_1 := 61/11$

$x_{m+1} := 111 - (1130\text{-}3000/x_{m-1})/x_m \to \mathbf{6}$

Models of *Real* Computation

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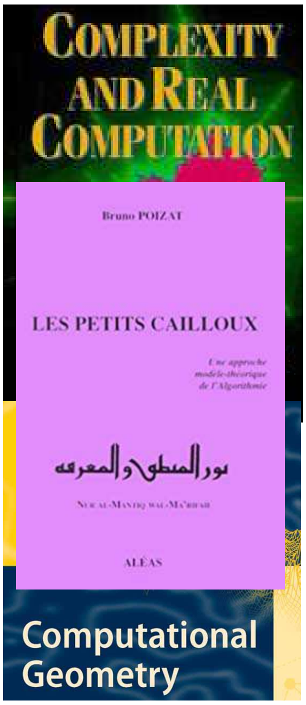
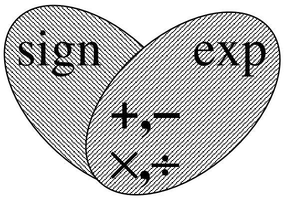
ℝ, not ℚ nor 𝔸

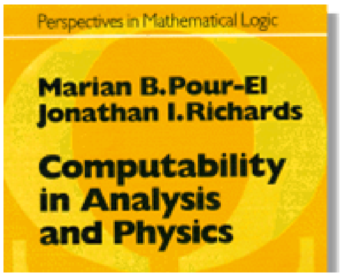
[Turing'37], [Grzegorczyk'57]

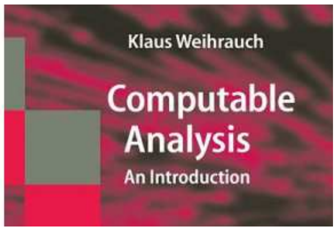




realPCF
[Escardó'96]
(functional)

THIS TALK
(imperative)





Imperative,
+,−, ×, ÷, <

<u>exact</u>

Computing by
streams of approximations

<u>Intermission</u>

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Floating point numbers (IEEE 754 from 1985)
are a hardware-based, *legacy* data type — like **`byte`**:

to be *hidden* behind object-oriented *ideal* data types!

Project: Develop <u>*continuous* abstract data types</u>.
- Fix some continuous structure (example: real numbers).
- Investigate computability of its operations/predicates. (Turing machines, *Computable Analysis*)
- Modify their semantics to *make them computable* and s.t.
- computing *over* this structure becomes Turing-*complete.*

Logic of Computing with Continuous Data:
Foundations of Numerical Software Engineering

Computing Real Numbers

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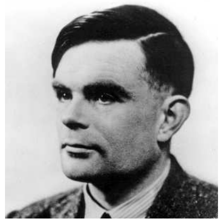

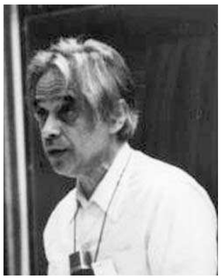

Fact: For $r\in\mathbb{R}$, the following are equivalent:

Call $r\in\mathbb{R}$ computable if

a) r has a recursive binary expansion

b) There exists a Turing machine printing a sequence $(a_n)\subseteq\mathbb{Z}$ with $|r-a/2^n|\leq 2^{-n}$.

c) There exists a Turing machine printing sequences $(a_n),(b_n)\subseteq\mathbb{Z}$ with $|r-a_n/b_n|\leq 1/n$

Ernst Specker (1949): (c)$^{\text{Halting problem}}$ $\Leftrightarrow$ (d)

d) There exists a machine printing $(q_n)\subseteq\mathbb{Q}$ with $q_n\to r$.

$\mathbb{R}$ as Computable Abstract Data Type

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- Fix some continuous structure (example: real numbers).
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Reminder: +, -, ×, ÷ are computable

Fact: There is a *computable* sequence $(r_j)\subseteq[0,1]$ such that $\{\,j : r_j\neq 0\,\} = H$, the Halting problem.

Non-extensional *„Parallel-OR"* [Escardó/Hofmann/Streicher'04]

- +, -, ×, ÷, 0, 1: *exact* operations
- *Partial* semantics of tests: „$x>y$" = ↑ if $x=y$.
- **choose**$(x_1>y_1,\ldots x_d>y_d)$ = *any* j s.t. $x_j>y_j$

provided it exists

Programming over Computable ℝ

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Example 0: *Fuzzy/soft* test [Chee K. Yap, …]

choose($y+2^{-n} > x$, $x+2^{-n} > y$)

```
REAL Trisection(INTEGER p; REAL→REAL f);
REAL x:=0; REAL y:=1;
WHILE ( choose(2^p>y−x , y−x>2^(p−1)) == 2)
 IF (choose(0>f((2x+y)/3),f((x+2y)/3)>0)==1)
  THEN  x:=(2x+y)/3  ELSE  y:=(x+2y)/3  ENDIF
```

$(\mathbb{Z},0,1,+,>)$ and $(\mathbb{R},0,1,+,\times,>)$ with $\iota:\mathbb{Z}\ni p\to 2^p\in\mathbb{R}$

Arguments *exact*, value *approximate* to 2^p.

- *Partial* semantics of tests: „**x>y**" = ↑ if $x=y$.
- **choose**($x_1>y_1,\ldots x_d>y_d$) = *any* j s.t. $x_j>y_j$

Logic of Computing over ℝ

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Theorem: a) Computing over the below two-sorted structure is Turing-*complete* for real functions!

Reconcile Blum-Shub-Smale model & Computable Analysis

b) Said two-sorted structure has *decidable* First-Order Theory!

⇐ [van den Dries'86]
→ formal verification.

c) Generalizing (a) to real function*al*s.

Also "*model complete*"
Wrong for $\iota(n)=1/n$

$(\mathbb{Z},0,1,+,>)$ and $(\mathbb{R},0,1,+,\times,>)$ with $\iota:\mathbb{Z}\ni p\to 2^p\in\mathbb{R}$

Arguments *exact*, value *approximate* to 2^p.

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Computing over Continuous Structures

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Theorem: a) Computing over the below two-sorted structure is Turing-*complete* for real functions!

Reconcile Blum-Shub-Smale model & Computable Analysis

b) Said two-sorted structure has *decidable* First-Order Theory! ⇐ [van den Dries'86] → formal verification.

c) Generalizing (a) to real functiona***l***s. Also "*model complete*" Wrong for $\iota(n)=1/n$

Continuous Abstract Data Types *in progress*:

- Compact Metric Groups
- Grassmannian
- Analytic Functions
- Compact Manifolds

Intermission

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Complexity of Computing with Continuous Data

Project: Develop *continuous* abstract data types.

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Logic of Computing with Continuous Data: Foundations of Numerical Software Engineering

Complexity Theory for Numerics

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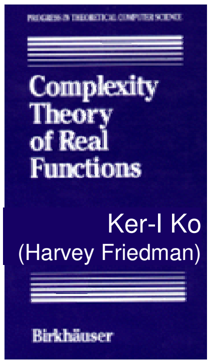
Fix polytime $f:[0;1]\to[0;1]$ ($\Rightarrow$ continuous)



- Max: $f \to \mathrm{Max}(f): x \to \max\{ f(t): t \le x \}$
  $\mathrm{Max}(f)$ computable in $\mathcal{PSPACE}$
  polyn.time-computable **iff** $\mathcal{P}=\mathcal{NP}$
- $\int: f \to \int f: (x \to \int_0^x f(t)\,dt)$ **even** for $f\in C^\infty$
  $\int f$ computable in $\mathcal{PSPACE}$
  polyn.time-computable **iff** $\mathcal{FP}=\#\mathcal{P}$

Polyn.time for analytic $f\in C^\omega$

- odesolve: $C^1([0;1]\times[-1;1])\ni f\to z$: $\dot z(t)=f(t,z)$, $z(0)=0$.
  $\mathcal{PSPACE}$-"complete" [Kawamura 2010]
- Solution to Poisson's Equation is classical and $\#\mathcal{P}$-"complete"

$$\Delta u = f \text{ on } B_2(\mathbf{0},1)$$
$$u = 0 \text{ on } \partial B_2(\mathbf{0},1)$$

[Kawamura+Steinberg+Z., MSCS 2017

Advanced (Complexity) Issues

Martin Ziegler

- Weihrauch Reduction among *continuous* problems

  2nd-order polynomial-time [Kawamura&Cook'2012]

- "Efficiently" encoding ("*representing*") spaces beyond $\mathbb{R}$
  such as Sobolev spaces (theory of PDEs)
- Computational complexity theory of *higher* types
- *Randomization* in continuous data processing
- *Formal verification* of *continuous* abstract data types
- *Software Testing* for *continuous* abstract data types

Logic of Computing with *Continuous* Data: Foundation of Numerical Software Engineering

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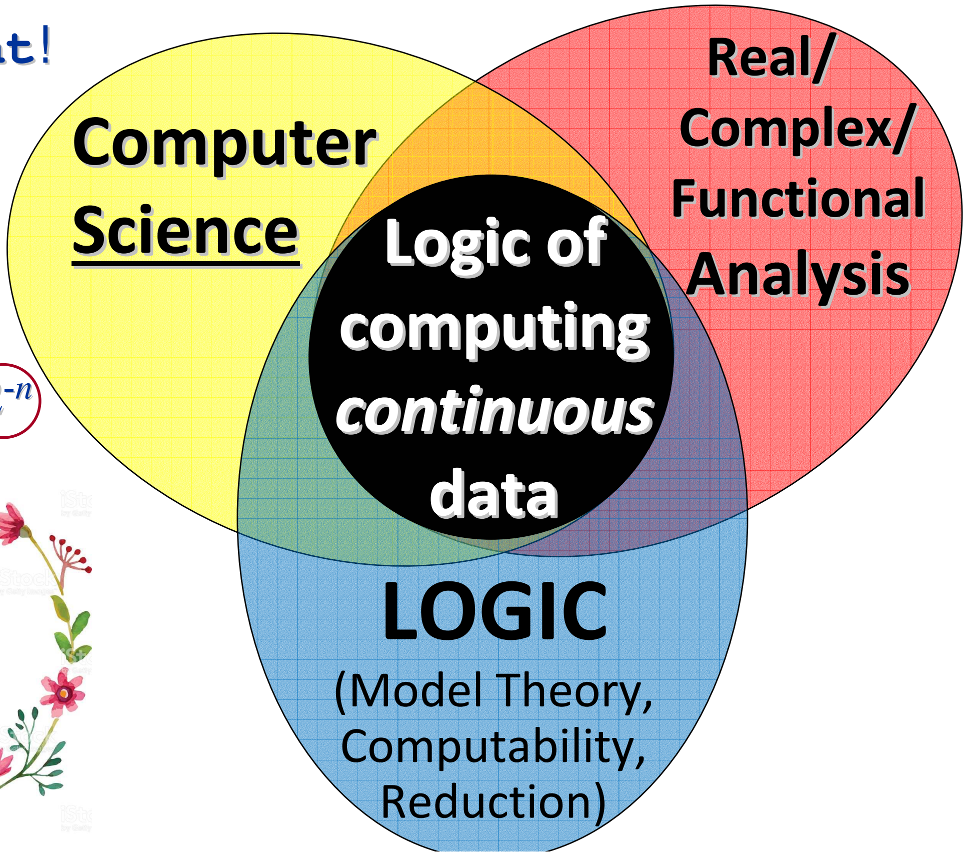
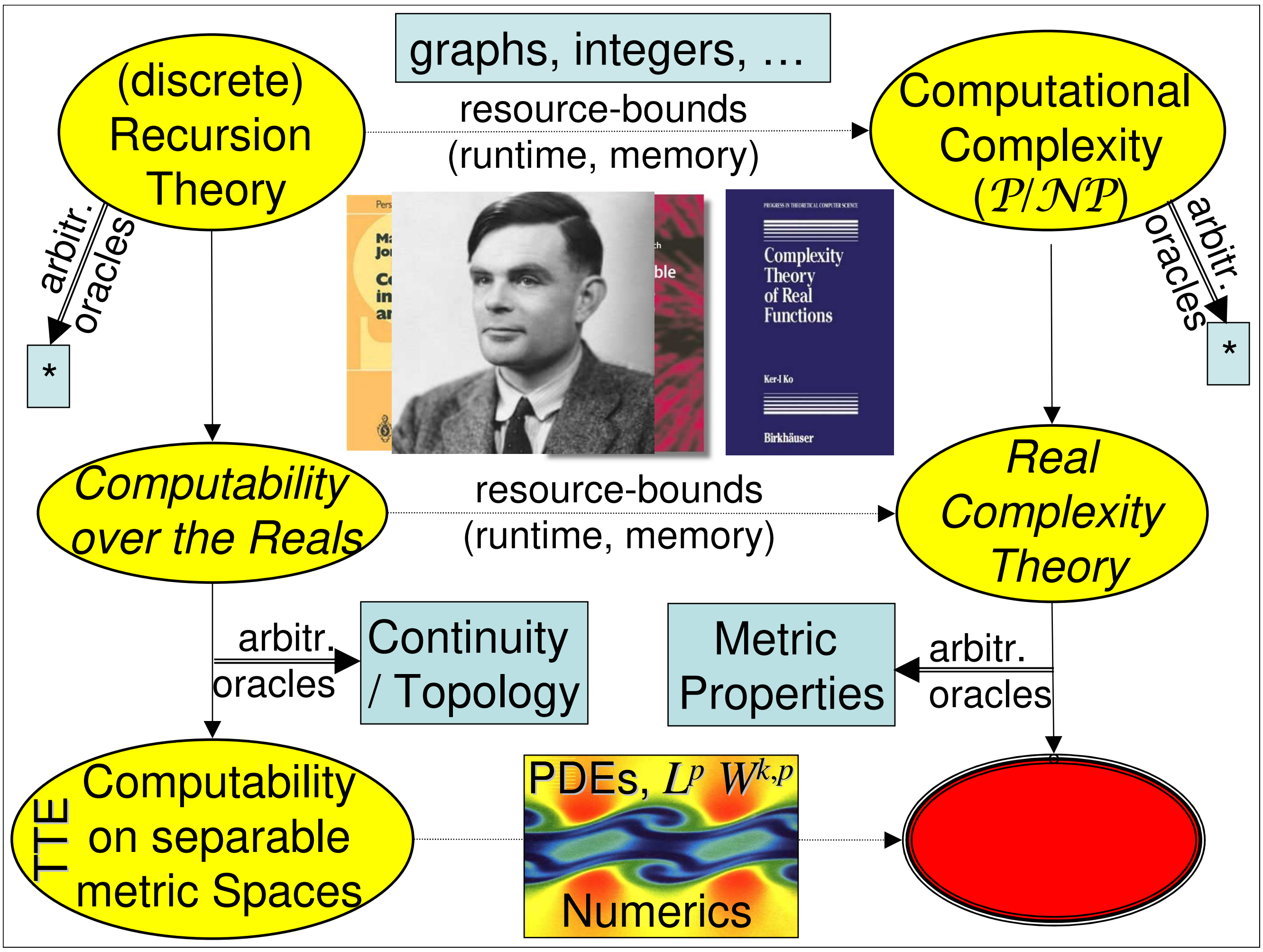
1. Forget `float`!
2. *Continuous* abstract data types
3. Complexity: approx. up to 2^{-n}





Foundations of Software Engineering

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discrete (integers, graphs, …)

<u>**Implementation/library**</u>

Abstract Data Types

Semantics/Verification
(Scott, Hoare, …)

Complexity
(Hartmanis, Cook, …)

Computability
(Turing, Kleene, …)

Specification

real (numbers, functions, operators)

continuous structures

„A good ***definition*** *is worth 1000 theorems“*
(Doron Zeilberger, Opinion #82)

Data Abstraction and Properties

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*In*completeness
(Gödel,Robinson)

logic.complete r.e.
1st-order axioms

algebraically complete

distributive law
$x \cdot (y+z) =$
$= x \cdot y + x \cdot z$

Algebraic Geometry

metrically complete

Existence of solutions to PDEs

ring, field

(finite) sets $\mathbb{N}$ $\mathbb{Z}$ $\mathbb{Q}$ $\mathbb{A}$ $\mathbb{R}$ $\mathbb{C}$ $\mathbb{P}$ $\mathbb{H}$ $\mathbb{W}$

`{0,1}* byte int` `double` *grids* of **`double`**, ad-hoc

abstract data type **`BigInteger`** (Java) **`iRRAM:REAL`** **Xerox PARC**

Disambiguation

Martin Ziegler

- *Numerical* Software Engineering
- *Continuous* Abstract Data Types
- Closest *agreement* with *structures* in Logic
- Rigorous elegant *computable* specification
- *without* hassles of *actual* Turing machines.
- *Imperative* Turing-*complete* programming
- Include transcendental computation
- Guided by Newton's Method: arguments exact, result approx.