Describing Countable Structures

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How would you describe the group $\mathbb{Q}$ uniquely up to isomorphism?

It is the rank 1 divisible torsion-free abelian group.

- How complicated is this description?
- Is there a simpler description?

Outline

1. Scott sentence complexity
2. Computable structures of high Scott sentence complexity
3. Finitely generated structures and other algebraic structures

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We write down descriptions in the logic $\mathcal{L}_{\omega_1\omega}$. Formulas are built using:

- equalities and inequalities of terms,
- relations,
- the connectives $\wedge$, $\vee$, and $\neg$,
- the quantifiers $\exists x$ and $\forall x$.
- the countably infinite connectives $\bigwedge$ and $\bigvee$.

The property of being a *rank 1 divisible torsion-free abelian group* can be expressed in $\mathcal{L}_{\omega_1\omega}$:

- group axioms, e.g.:

$$(\forall x)\ \ x + 0 = 0 + x = x$$

- abelian:

$$(\forall x \forall y)\ \ x + y = y + x$$

- torsion-free:

$$(\forall x \neq 0)\ \bigwedge_{n \geq 1} nx \neq 0$$

- rank 1:

$$(\forall x \forall y)\ \bigvee_{(n,m) \neq (0,0)} nx = my$$

- divisible:

$$(\forall x)\ \bigwedge_{n \geq 1} (\exists y)\ x = ny$$

Infinitary logic is expressive enough to describe every countable structure.

Theorem (Scott 1965)

For every countable structure $\mathcal{A}$, there is an $\mathcal{L}_{\omega_1\omega}$ formula φ such that $\mathcal{A}$ is the only countable structure satisfying φ.

We call any such sentence a Scott sentence for $\mathcal{A}$.

Main Idea *Measure the complexity of a structure by the complexity of the simplest Scott sentence for that structure.*

We can define a hierarchy of $\mathcal{L}_{\omega_1\omega}$-formulas based on their quantifier complexity after putting them in normal form.

- A formula is Σ_0 and Π_0 is it is finitary quantifier-free.
- A formula is Σ_α if it looks like

$$\bigvee_{n\in\mathbb{N}} (\exists \bar{x})\varphi_n$$

  where the φ are Π_β for $\beta < \alpha$.
- A formula is Π_α if it looks like

$$\bigwedge_{n\in\mathbb{N}} (\forall \bar{x})\varphi_n$$

  where the φ are Σ_β for $\beta < \alpha$.

The vector space $\mathbb{Q}^{\mathbb{N}}$ has a Π_3 Scott sentence. We say that it is infinite-dimensional as follows:

$$\underbrace{\bigwedge_{n \in \mathbb{N}} \underbrace{(\exists x_1, \ldots, x_n) \underbrace{\bigwedge_{c_1, \ldots, c_n \in \mathbb{Q}} \underbrace{\left[c_1 x_1 + \cdots + c_n x_n = 0 \to \left[c_1 = c_2 = \cdots = c_n = 0 \right] \right]}_{\Sigma_0}}_{\Pi_1}}_{\Sigma_2}}_{\Pi_3}.$$

The property of being a *rank 1 divisible torsion-free abelian group* can be expressed in $\mathcal{L}_{\omega_1\omega}$:

- group axioms, e.g.:

$$(\forall x)\ \ x+0=0+x=x \qquad (\Pi_1)$$

- abelian:

$$(\forall x \forall y)\ \ x+y=y+x \qquad (\Pi_1)$$

- torsion-free:

$$(\forall x\neq 0)\ \bigwedge_{n\geq 1} nx\neq 0 \qquad (\Pi_1)$$

- rank 1:

$$(\forall x \forall y)\ \bigvee_{(n,m)\neq(0,0)} nx=my \qquad (\Pi_2)$$

- divisible:

$$(\forall x)\ \bigwedge_{n\geq 1}(\exists y)\ x=ny \qquad (\Pi_2)$$

The group $\mathbb{Q}$ has a Π_2 Scott sentence.

One way of measuring the complexity of a structure is its Scott rank. Many different definitions of Scott rank have been put forward. They are almost, but not quite, equivalent. One is:

Definition (Montalbán)

The Scott rank of $\mathcal{A}$ is the least ordinal α such that $\mathcal{A}$ has a $\Pi_{\alpha+1}$ Scott sentence.

This is a robust notion of complexity.

Theorem (Montalbán)

Let $\mathcal{A}$ be a countable structure and let α a countable ordinal. The following are equivalent:

- *$\mathcal{A}$ has a $\Pi_{\alpha+1}$ Scott sentence.*
- *Every automorphism orbit in $\mathcal{A}$ is Σ_α-definable without parameters.*
- *$\mathcal{A}$ is uniformly (boldface) $\boldsymbol{\Delta}^0_\alpha$-categorical without parameters.*

A Scott sentence for the group $\mathbb{Z}$ consists of:

- the axioms for torsion-free abelian groups,
- for any two elements, there is an element which generates both,
- there is a non-zero element with no proper divisors:

$$(\exists g \neq 0) \bigwedge_{n \geq 2} (\forall h)[nh \neq g].$$

These are, respectively, Π_1, Π_2, and Σ_2. So the Scott sentence is the conjunction of a Π_2 sentence and a Σ_2 sentence.

The Scott rank of $\mathbb{Z}$ is 2, the same as the vector space $\mathbb{Q}^{\mathbb{N}}$, even though $\mathbb{Z}$ has a simpler Scott sentence.

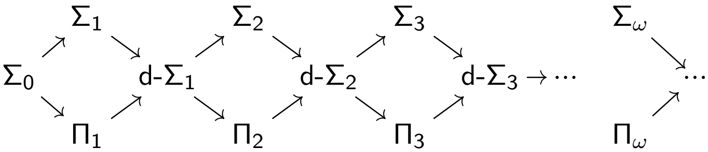
Scott rank does not make all the distinctions that we want it to; we need a finer notion.

Definition

A formula is d-Σ_α if it is the conjunction of a Σ_α formula and a Π_α formula.

So the group $\mathbb{Z}$ has a d-Σ_2 Scott sentence.

The picture we have now looks like:



This is not a complete picture; there are other possible complexities.

We want to make the following definition, but we have not been able to say formally what a "complexity" of a sentence is.

Definition

The Scott sentence complexity of a countable structure $\mathcal{A}$ is the least complexity of a Scott sentence for $\mathcal{A}$.

There are some restrictions on the possible Scott complexities of structures.

For example, Σ_ω is not a possible Scott sentence complexity: Suppose $\mathcal{A}$ has a Σ_ω Scott sentence

$$\varphi_1 \vee \varphi_2 \vee \varphi_3 \vee \varphi_4 \vee \cdots$$

where each φ_i is Σ_n for some n. For some i, $\mathcal{A} \models \varphi_i$. Then φ_i is a Σ_n Scott sentence for $\mathcal{A}$.

A deeper theorem is:

Theorem (A. Miller)

Let $\mathcal{A}$ be a countable structure. If $\mathcal{A}$ has a $\Sigma_{\alpha+1}$ Scott sentence, and also has a $\Pi_{\alpha+1}$ Scott sentence, then $\mathcal{A}$ has a d-Σ_α Scott sentence.

To make this more formal, we turn to Wadge degrees.

Fix a language $\mathcal{L}$, and for simplicity assume that $\mathcal{L}$ is relational. We can view the space of $\mathcal{L}$-structures with domain ω as a Polish space isomorphic to Cantor space 2^ω. Call this $\mathrm{Mod}(\mathcal{L})$.

E.g., if $\mathcal{L} = \{R\}$ with R unary, associate to an $\mathcal{L}$-structure $\mathcal{M} = (\omega, R^{\mathcal{M}})$ the element $\alpha \in 2^\omega$ with

$$\alpha(n) = \begin{cases} 0 & n \notin R^{\mathcal{M}} \\ 1 & n \in R^{\mathcal{M}} \end{cases}$$

Lopez-Escobar proved a powerful theorem relating $\mathcal{L}_{\omega_1\omega}$ classes and Borel sets in $\mathrm{Mod}(\mathcal{L})$. Vaught proved a level-by-level version of this theorem:

Theorem (Vaught)

Let $\mathbb{K}$ *be a subclass of* $\mathrm{Mod}(\mathcal{L})$ *which is closed under isomorphism.*

$\mathbb{K}$ *is* $\mathbf{\Sigma}^0_\alpha$ *in the Borel hierarchy.*

$$\Updownarrow$$

$\mathbb{K}$ *is axiomatized by an infinitary* Σ_α *sentence.*

The same is true for $\mathbf{\Pi}^0_\alpha$ and Π_α, the Ershov hierarchy (including d-Σ_α), etc.

We measure the complexity of subsets of $\mathrm{Mod}(\mathcal{L})$ using the Wadge hierarchy.

Definition (Wadge)

Let A and B be subsets of Cantor space 2^ω. We say that A is *Wadge reducible* to B, and write $A \leq_W B$, if there is a continuous function f on 2^ω with $A = f^{-1}[B]$, i.e.

$$x \in A \Longleftrightarrow f(x) \in B.$$

The Wadge hierarchy has a lot of structure.

Theorem (Martin and Monk, AD)

The Wadge order is well-founded.

Theorem (Wadge's Lemma, AD)

Given $A, B \subseteq \omega^\omega$, either $A \leq_W B$ or $B \leq_W \omega^\omega - A$.

Given a countable structure $\mathcal{A}$, let $\mathrm{Iso}(\mathcal{A})$ be the set of isomorphic copies of $\mathcal{A}$ in $\mathrm{Mod}(\mathcal{L})$.

By the Lopez-Escobar theorem, informally we see that the complexity of Scott sentences for $\mathcal{A}$ corresponds to the location of $\mathrm{Iso}(\mathcal{A})$ in the Wadge hierarchy.

Definition

The *Scott sentence complexity* of a structure $\mathcal{A}$ is the Wadge degree of $\mathrm{Iso}(\mathcal{A})$.

Theorem (A. Miller 1983, Alvir-Greenberg-HT-Turetsky)

The possible Scott complexities of countable structures $\mathcal{A}$ are:

1. Π_α *for* $\alpha \geq 1$,
2. Σ_α *for* $\alpha \geq 3$ *a successor ordinal,*
3. $\mathrm{d}\text{-}\Sigma^0_\alpha$ *for* $\alpha \geq 1$ *a successor ordinal.*

There is a countable structure with each of these Wadge degrees.

Wadge's Lemma is the key ingredient to narrow it down to these possibilities.

That Σ_2 is not possible was shown by A. Miller for relational structures and by Alvir-Greenberg-HT-Turetsky for general structures.

A. Miller constructed examples of most of these except for $\Sigma_{\lambda+1}$ for λ a limit ordinal; these were constructed by Alvir-Greenberg-HT-Turetsky.

Proposition (Montalbán, Alvir-Greenberg-HT-Turetsky)

Let $\mathcal{A}$ be a countable structure. Then:

1. *$\mathcal{A}$ has a $\Sigma_{\alpha+1}$ Scott sentence if and only if for some $\bar{c} \in \mathcal{A}$, $(\mathcal{A}, \bar{c})$ has a Π_α Scott sentence.*
2. *$\mathcal{A}$ has a $\mathrm{d}\text{–}\Sigma_\alpha$ Scott sentence if and only if for some $\bar{c} \in \mathcal{A}$, $(\mathcal{A}, \bar{c})$ has a Π_α Scott sentence and the automorphism orbit of $\bar{c}$ is Σ_α-definable.*

Theorem (Montalbán)

Let α a countable ordinal. The following are equivalent:

- *$\mathcal{A}$ has a $\Sigma^0_{\alpha+2}$ Scott sentence.*
- *There are parameters over which every automorphism orbit in $\mathcal{A}$ is Σ^0_α-definable.*
- *$\mathcal{A}$ is relatively (boldface) $\boldsymbol{\Delta}^0_\alpha$-categorical*

Outline

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A computable structure is a structure with domain ω all of whose relations and functions are uniformly computable.

A computable $\mathcal{L}_{\omega_1\omega}$ formula is one in which all of the infinitary conjunctions and disjunctions are effective.

The ordinal ω_1^{CK} is the least ordinal which is not computable. (Given $x \in 2^\omega$, ω_1^x is the least ordinal which is not x-computable.)

Every computable $\mathcal{L}_{\omega_1\omega}$ formula is Σ_α for some $\alpha < \omega_1^{CK}$.

Nadel analysed the Scott sentences of computable structures.

Theorem (Nadel 1974)

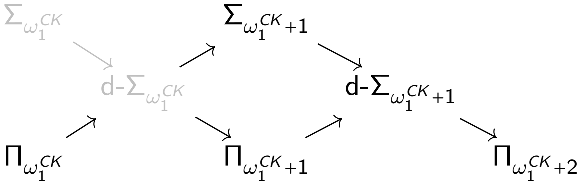
- *Every computable structure has a $\Pi_{\omega_1^{CK}+2}$ Scott sentence.*
- *A computable structure has a computable Scott sentence if and only if it has Scott sentence complexity strictly less than $\Pi_{\omega_1^{CK}}$.*

We say that a structure has high Scott sentence complexity / high Scott rank if it has Scott sentence complexity $\Pi_{\omega_1^{CK}}$ or higher / Scott rank ω_1^{CK} or higher.

A structure has low Scott sentence complexity if and only if it has a computable Scott sentence.

Until recently we thought of structures of high Scott rank as being divided into two possible ranks: ω_1^{CK} and $\omega_1^{CK}+1$.

Now, there are five possible Scott sentence complexities for a computable structure of high Scott sentence complexity.



Scott rank ω_1^{CK}	:	$\Pi_{\omega_1^{CK}}$, $\Pi_{\omega_1^{CK}}+1$
Scott rank $\omega_1^{CK}+1$	:	$\Sigma_{\omega_1^{CK}+1}$, d-$\Sigma_{\omega_1^{CK}+1}$, $\Pi_{\omega_1^{CK}+2}$

For a computable structure, having a Scott sentence of the form on the left is equivalent to the condition on the right:

$\Pi_{\omega_1^{CK}}$	:	computable infinitary theory is $\aleph_0$-categorical.
$\Pi_{\omega_1^{CK}} + 1$	:	each automorphism orbit is definable by a computable formula.
$\Sigma_{\omega_1^{CK}+1}$	:	after naming constants, computable infinitary theory is $\aleph_0$-categorical.
d-$\Sigma_{\omega_1^{CK}+1}$	:	each automorphism orbit is definable by a computable formula over parameters which are $\Sigma_{\omega_1^{CK}+1}$-definable.
$\Pi_{\omega_1^{CK}+2}$	:	always.

The first structure of high Scott sentence complexity was constructed by Harrison.

Theorem (Harrison)

There is a computable order of order type $\omega_1^{CK} \cdot (1 + \mathbb{Q})$*. This has Scott sentence complexity* $\Pi_{\omega_1^{CK}+2}$*.*

The Harrison linear order is natural: There is a computable operator $x \mapsto H_x$ such that H_x is a linear order of order type $\omega_1^x(1 + \mathbb{Q})$.

In a sense, all natural structure of high Scott sentence complexity have Scott sentence complexity $\Pi_{\omega_1^{CK}+2}$.

Theorem (Becker, Chan-HT-Marks)

If $x \mapsto \mathcal{A}_x$ is a Borel operator such that

$$\omega_1^x = \omega_1^y \Longrightarrow \mathcal{A}_x \cong \mathcal{A}_y$$

then for some x, $\mathcal{A}_x$ has Scott sentence complexity $\Pi_{\omega_1^x+2}$.

The second type of structure of high Scott sentence complexity was constructed by Makkai, Knight, and Millar.

Theorem (Makkai, Knight-Millar)

There is a computable structure of Scott sentence complexity $\Pi_{\omega_1^{CK}}$.

The computable infinitary theory of such a structure is $\aleph_0$-categorical.

Millar and Sacks asked whether there is a computable structure of Scott rank ω_1^{CK} whose computable infinitary theory is not $\aleph_0$-categorical. (Millar and Sacks had produced such a structure which was not computable, but which had $\omega_1^{\mathcal{A}} = \omega_1^{CK}$.)

This is exactly the same as asking for a computable structure of Scott sentence complexity $\Pi_{\omega_1^{CK}+1}$.

Theorem (HT-Igusa-Knight)

There is a computable structure of Scott sentence complexity $\Pi_{\omega_1^{CK}+1}$.

Another open question was whether there is a computable structure of Scott rank $\omega_1^{CK} + 1$ which has Scott rank ω_1^{CK} after naming constants.

It turned out that this is the same as asking for a computable structure of Scott sentence complexity $\Sigma_{\omega_1^{CK}+1}$ or d-$\Sigma_{\omega_1^{CK}+1}$.

Theorem (Alvir-Greenberg-HT-Turetsky)

There are computable structures of Scott sentence complexity $\Sigma_{\omega_1^{CK}+1}$ and d-$\Sigma_{\omega_1^{CK}+1}$.

The possible Scott complexities were:

$$\begin{array}{ccccccccc}
\Sigma_{\omega_1^{CK}} & & & & \Sigma_{\omega_1^{CK}+1} & & & & \\
& \searrow & & \nearrow & & \searrow & & & \\
& & \text{d-}\Sigma_{\omega_1^{CK}} & & & & \text{d-}\Sigma_{\omega_1^{CK}+1} & & \\
& \nearrow & & \searrow & & \nearrow & & \searrow & \\
\Pi_{\omega_1^{CK}} & & & & \Pi_{\omega_1^{CK}+1} & & & & \Pi_{\omega_1^{CK}+2}
\end{array}$$

We have examples of all of these!

There are some other kinds of examples of structure of high Scott sentence complexity.

Theorem (HT)

There is a computable structure $\mathcal{A}$ of high Scott sentence complexity which is not computably approximable. (There is a Π_2 sentence φ true of $\mathcal{A}$ such that every model of φ has high Scott sentence complexity.)

Theorem (Turetsky)

There is a computably categorical structure of high Scott sentence complexity.

These are properties none of the other examples had.

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Recall that a structure is finitely generated if there is a finite tuple $\bar{a}$ of elements such that every element is the image of $\bar{a}$ under some composition of functions.

Theorem (Knight-Saraph)

Every finitely generated structure has a Σ_3 Scott sentence.

Often there is a simpler Scott sentence; we have already seen the example of the group $\mathbb{Z}$, which has a d-Σ_2 Scott sentence.

It seemed like most finitely generated groups have a d-Σ_2 Scott sentence.

Theorem (Knight-Saraph, CHKLMMMQW, Ho)

The following groups all have d-Σ_2 Scott sentences:

- *abelian groups,*
- *free groups,*
- *nilpotent groups,*
- *polycyclic groups,*
- *lamplighter groups,*
- *Baumslag-Solitar groups $BS(1, n)$.*

Knight asked: Is this always the case?

Theorem (A. Miller, HT-Ho, Alvir-Knight-McCoy)

Let $\mathcal{A}$ be a finitely generated structure. The following are equivalent:

- *$\mathcal{A}$ has a Π_3 Scott sentence.*
- *$\mathcal{A}$ has a d-Σ_2 Scott sentence.*
- *$\mathcal{A}$ is the only model of its Σ_2 theory.*
- *some generating tuple of $\mathcal{A}$ is defined by a Π_1 formula.*
- *every generating tuple of $\mathcal{A}$ is defined by a Π_1 formula.*
- *$\mathcal{A}$ does not contain a copy of itself as a proper Σ_1-elementary substructure.*

Theorem (HT-Ho)

There is a finitely generated group G which does not have a d-Σ_2 Scott sentence.

Open Question

Does every finitely presented group have a d-Σ_2 Scott sentence?

Theorem (HT)

A random finitely presented group has a d-Σ_2 Scott sentence.

Theorem (HT)

Every finitely generated commutative ring has a d-Σ_2 Scott sentence.

Simple classes; every structure has a d-Σ_2 Scott sentence:

- abelian groups, (Knight-Saraph)
- free groups, (CHKLMMMQW)
- torsion-free hyperbolic groups, (HT)
- vector spaces, (Folklore)
- fields, (HT-Ho)
- commutative rings, (HT)
- modules over Noetherian rings. (HT)

Complicated classes; there is a structure with no d-Σ_2 Scott sentence:

- groups, (HT-Ho)
- rings, (HT-Ho)
- modules. (HT)

The Nielson transformations give us a very good understanding of bases for free groups.

Theorem (CHKLMMMQW)

The free group on countably many generators has a Π_4 Scott sentence.

We do not have such an understanding for pure transcendental fields.

Open Question

What is the Scott sentence complexity of the field $\mathbb{Q}(x_1, x_2, \ldots)$?

Thanks!