The ultrafilter and almost disjointness numbers

Osvaldo Guzmán joint work with Damjan Kalajdzievski

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The *cardinal invariants of the continuum* are uncountable cardinals whose size is at most the cardinality of the real numbers. We are mostly interested in cardinals with a nice topological or combinatorial definition.

1. By \( \omega \) we denote the set (cardinal) of the natural numbers.
2. By \( c \) we denote the cardinality of the real numbers.
The cardinal invariants of the continuum are cardinals $j$ such that:

$$\omega < j \leq c$$

The Continuum Hypothesis ($CH$) is the following statement:

$c$ is the first uncountable cardinal

All cardinal invariants are $c$ under $CH$.

Martin’s Axiom (MA) implies that most cardinal invariants are $c$. 
We are interested in studying the relationships between different cardinal invariants.
a  almost disjointness number
b  bounding number
c  cardinality of the continuum
d  dominating number
e  evasion number
f  free sequence number
g  groupwise number
h  distributivity number
i  independence number
j
k
l  Laver property number
m  Martin’s number
n  Novak’s number (might be bigger than c)
o  the offbranch number
p pseudointersection number
q Q-set number
r reaping number
s splitting number
t tower number
u ultrafilter number
v
w
x
y
z sequence number

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The ultrafilter and almost disjointness number
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\( h_{om} \) ss \( a_g \)
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An infinite family \( \mathcal{A} \subseteq [\omega]^\omega \) is \textit{almost disjoint (AD)} if the intersection of any two different elements of \( \mathcal{A} \) is finite. A MAD \textit{family} is a maximal almost disjoint family.

The \textit{almost disjointness number} \( \alpha \) is the smallest size of a MAD family.
We say that a family $\mathcal{F} \subseteq \wp(\omega)$ is a filter\(^1\) if the following conditions hold:

1. $\omega \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$.
2. If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$.
3. If $A \in \mathcal{F}$ and $A \subseteq B$ then $B \in \mathcal{F}$.
4. $\mathcal{F} \cap [\omega]^{<\omega} = \emptyset$.

The concept of a filter formalizes a kind of “largeness” notion, the elements which belong to the filter are regarded as large, while its complements are regarded as small. An ultraliter is a maximal filter.

\(^1\)By $\omega$ we denote the set of natural numbers.
1 In the same way as with MAD families, we could define an invariant as “the smallest size of an ultrafilter” but this invariant will be \( c \).

2 We need the notion of an *ultrafilter base*.
Definition

We say that \( \mathcal{B} \subseteq [\omega]^{\omega} \) is an *ultrafilter base* if the set
\[
\{ A \mid \exists B \in \mathcal{B} (B \subseteq A) \}
\]
is an ultrafilter.

1. The *ultrafilter number* \( u \) denotes the smallest size of a base for an ultrafilter on \( \omega \).
\( \alpha \) the smallest size of a MAD family

\( \mu \) the smallest size of a base for an ultrafilter on \( \omega \).

1. \( \alpha \) and \( \mu \) are cardinal invariants.
2. \( \omega \leq \alpha, \mu \leq c \).
3. What is the relationship between them?
If we assume the Continuum Hypothesis, then $\omega_1 = a = u = c$. 
1. If we assume the Continuum Hypothesis, then $\omega_1 = \mathfrak{a} = \mathfrak{u} = \mathfrak{c}$.

2. The consistency of the inequality $\mathfrak{a} < \mathfrak{u}$ is well known and easy to prove, in fact, it holds in the Cohen, random and Silver models, among many others.
1. If we assume the Continuum Hypothesis, then $\omega_1 = a = u = c$.

2. The consistency of the inequality $a < u$ is well known and easy to prove, in fact, it holds in the Cohen, random and Silver models, among many others.

3. Proving the consistency of the inequality $u < a$ is much harder and used to be an open problem for a long time. In fact, it follows by the theorems of Hrušák, Moore and Džamonja that the inequality $u < a$ can not be obtained by using countable support iteration of proper Borel partial orders.
The consistency of $\mu < \alpha$ was finally established by Shelah, when he proved the following theorem:

**Theorem (Shelah)**

Let $V$ be a model of GCH, $\kappa$ a measurable cardinal and $\mu, \lambda$ two regular cardinals such that $\kappa < \mu < \lambda$. There is a c.c.c. forcing extension of $V$ that satisfies $\mu = \mathfrak{b} = \mathfrak{d} = \mathfrak{u}$ and $\lambda = \mathfrak{a} = \mathfrak{c}$. In particular, $\text{CON}(\text{ZFC} + \text{“there is a measurable cardinal”})$ implies $\text{CON}(\text{ZFC} + \text{“$\mu < \alpha$”})$. 
**Theorem (Shelah)**

Let $V$ be a model of GCH, $\kappa$ a measurable cardinal. There is a c.c.c. forcing extension of $V$ that satisfies $\mathfrak{u} = \kappa^+$ and $\alpha = \mathfrak{c} = \kappa^{++}$. In particular, $\text{CON}(\text{ZFC} + \text{there is a measurable cardinal})$ implies $\text{CON}(\text{ZFC} + \mathfrak{u} < \alpha)$.

This theorem was one of the first results proved using “template iterations”, which is a very powerful method that has been very useful and has been successfully applied to this day. In spite of the beauty of this result, it leaves open the following questions:

**Problem (Shelah)**

Does $\text{CON}(\text{ZFC})$ imply $\text{CON}(\text{ZFC} + \mathfrak{u} < \alpha)$?

**Problem (Brendle)**

Is it consistent that $\omega_1 = \mathfrak{u} < \alpha$?
With Damjan Kalajdzievski, we were able provide a positive answer to both questions, by proving (without appealing to large cardinals) that every MAD family can be destroyed by a proper forcing that preserves $P$-points.
The method of forcing consists of adding a new set to the universe, in a similar way as adding a new root to a field. Forcing extensions are performed using partial orders.

In our case, we want to add a new set that destroys the maximality of a given MAD family, while preserving an ultrafilter base (of a $P$-point).
Definition

Let $\mathbb{P}$ be a partial order, $\mathcal{F}$ a filter and $\mathcal{U}$ an ultrafilter.

1. $\mathbb{P}$ diagonalizes $\mathcal{F}$ if $\mathbb{P}$ adds an infinite set almost contained in every element of $\mathcal{F}$.

2. $\mathbb{P}$ preserves $\mathcal{U}$ if $\mathcal{U}$ is the base of an ultrafilter after forcing with $\mathbb{P}$.

There are two usual forcings for diagonalizing a filter.
Definition

The *Laver forcing* $\mathbb{L}(\mathcal{F})$ with respect to $\mathcal{F}$ is the set of all trees $p$ such that $\text{suc}_p(s) \in \mathcal{F}$ for every $s \in p$ extending the stem of $p$ (where $\text{suc}_p(s) = \{ n \mid s \upharpoonright n \in p \}$). We say $p \leq q$ if $p \subseteq q$.

Definition

If $\mathcal{F}$ is a filter on $\omega$ (or on any countable set) we define the *Mathias forcing* $\mathbb{M}(\mathcal{F})$ with respect to $\mathcal{F}$ as the set of all pairs $(s, A)$ where $s \in [\omega]^{<\omega}$ and $A \in \mathcal{F}$. If $(s, A), (t, B) \in \mathbb{M}(\mathcal{F})$ then $(s, A) \leq (t, B)$ if the following conditions hold:

1. $t$ is an initial segment of $s$.
2. $A \subseteq B$.
3. $(s \setminus t) \subseteq B$. 
1. Let \( f, g \in \omega^\omega \), define \( f \leq^* g \) if and only if \( f(n) \leq g(n) \) holds for all \( n \in \omega \) except finitely many. We say a family \( \mathcal{B} \subseteq \omega^\omega \) is unbounded if \( \mathcal{B} \) is unbounded with respect to \( \leq^* \).

2. The **bounding number** \( \mathfrak{b} \) is the size of the smallest unbounded family.

3. We say that \( S \) splits \( X \) if \( S \cap X \) and \( X \setminus S \) are both infinite. A family \( S \subseteq [\omega]^\omega \) is a **splitting family** if for every \( X \in [\omega]^\omega \) there is \( S \in S \) such that \( S \) splits \( X \).

4. The **splitting number** \( \mathfrak{s} \) is the smallest size of a splitting family.
It is not difficult to prove that $b \leq a$ and $b \leq u$.

Our model will be a model of $\omega_1 = b = u < a = s = \omega_2$. We will first explain how to build a model of $u < s$. 

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Theorem (Blass-Shelah)

The inequality \( u < s \) is consistent.

It is easy to see that diagonalizing an ultrafilter destroys all ground model splitting families. In this way, if we want to build a model of \( u < s \), we need to diagonalize an ultrafilter, while preserving another one (in fact, preserving a \( P \)-point). This topic has also been recently studied by Heike Mildenberger.
While $\mathbb{L}(\mathcal{F})$ always adds a dominating real, this may not be the case for $\mathbb{M}(\mathcal{F})$. A trivial example is taking $\mathcal{F}$ to be the cofinite filter in $\omega$, since in this case $\mathbb{M}(\mathcal{F})$ is forcing equivalent to Cohen forcing. A more interesting example was found by Canjar, where an ultrafilter whose Mathias forcing does not add dominating reals was constructed under $\mathfrak{d} = \mathfrak{c}$.

**Definition**

We say that a filter $\mathcal{F}$ is **Canjar** if $\mathbb{M}(\mathcal{F})$ does not add dominating reals.

In order to provide a combinatorial characterization of the previous notion, we need the following definition:
Definition

Let $\mathcal{F}$ be a filter on $\omega$. Define the filter $\mathcal{F}^{<\omega}$ in $[\omega]^{<\omega} \setminus \{\emptyset\}$ as the filter generated by $\{[A]^{<\omega} \setminus \{\emptyset\} \mid A \in \mathcal{F}\}$.

Note that if $X \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$, then $X \in (\mathcal{F}^{<\omega})^+$ if and only if for every $A \in \mathcal{F}$, there is $s \in X$ such that $s \subseteq A$. 
Theorem

Let \( \mathcal{F} \) be a filter on \( \omega \). The following are equivalent:

1. \( \mathcal{F} \) is Canjar.

2. (Hrušák, Minami) For every \( \{X_n \mid n \in \omega\} \subseteq (\mathcal{F}^{<\omega})^+ \) there are \( Y_n \in [X_n]^{<\omega} \) such that \( \bigcup_{n \in \omega} Y_n \in (\mathcal{F}^{<\omega})^+ \).

3. (Chodounský, Repovš and Zdomskyy) \( \mathcal{F} \) is Menger (as a subspace of \( \wp(\omega) \cong 2^\omega \)).\(^a\)

\(^a\)We view filters as subspaces of \( 2^\omega \), the notion of Borel or \( F_\sigma \) is taken using the usual topology on \( 2^\omega \).
Let $\mathcal{F}$ be a filter. The Canjar game $\mathcal{G}_{\text{Canjar}}(\mathcal{F})$ is defined as follows:

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<thead>
<tr>
<th>I</th>
<th>$X_0$</th>
<th>$X_1$</th>
<th>$X_2$</th>
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<tbody>
<tr>
<td>II</td>
<td>$Y_0$</td>
<td>$Y_1$</td>
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</tbody>
</table>

Where $X_i \in (\mathcal{F}^{<\omega})^+$ and $Y_i \in [X_i]^{<\omega}$ for every $i \in \omega$. The player II wins the game $\mathcal{G}_{\text{Canjar}}(\mathcal{F})$ if $\bigcup_{n \in \omega} Y_n \in (\mathcal{F}^{<\omega})^+$. 
Theorem (Chodounský, Repovš and Zdomskyy)

Let $\mathcal{F}$ be a filter on $\omega$. The following are equivalent:

1. $\mathcal{F}$ is Canjar.
2. Player I does not have a winning strategy in $G_{\text{Canjar}}(\mathcal{F})$. 
Definition

\( U \) is a P-point if every countable subfamily \( B \subseteq U \) there is \( A \in U \) such that \( A \setminus B \) is finite for every \( B \in B \).
Let $\mathcal{U}$ be an ultrafilter. Recall that the $P$-point game $G_{\text{P-point}}(\mathcal{U})$ is defined as follows:

<table>
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<tr>
<th>I</th>
<th>$W_0$</th>
<th>$W_1$</th>
<th>...</th>
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<tbody>
<tr>
<td>II</td>
<td>$z_0$</td>
<td>$z_1$</td>
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Where $W_i \in \mathcal{U}$ and $z_i \in [W_i]^{<\omega}$. The player II will win the game $G_{\text{P-point}}(\mathcal{U})$ if $\bigcup_{m \in \omega} z_m \in \mathcal{U}$. It is well known that player II can not have a winning strategy for this game. The following is a well known result of Galvin and Shelah:

**Theorem (Galvin-Shelah)**

Let $\mathcal{U}$ be an ultrafilter. $\mathcal{U}$ is a $P$-point if and only if Player I does not have a winning strategy in $G_{\text{P-point}}(\mathcal{U})$. 

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Let $G$ and $H$ be two (infinite) games:

<table>
<thead>
<tr>
<th>$G$</th>
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<tbody>
<tr>
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<td>II</td>
<td>$b_0$</td>
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<th>$H$</th>
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<tr>
<td></td>
<td>II</td>
<td>$d_0$</td>
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We define the game $G \ast H$ as follows:

<table>
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<tr>
<th>$G \ast H$</th>
<th>I</th>
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<tr>
<td></td>
<td>II</td>
<td>$b_0$</td>
<td>$d_0$</td>
<td>$b_1$</td>
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Where $\langle a_i, b_i \rangle_{i \in \omega}$ is played according to $G$ and $\langle c_i, d_i \rangle_{i \in \omega}$ is played according to $H$. Player II will win $G \ast H$ is $\langle a_i, b_i \rangle_{i \in \omega}$ is a winning run for Player II in $G$ and $\langle c_i, d_i \rangle_{i \in \omega}$ is a winning run for Player II in $H$. 

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Let $G$ and $H$ be two games. It seems obvious that if Player I does not have a winning strategy for $G$ or $H$, then he will not have a winning strategy for $G * H$ ... but this is false.

If $U$ is a $P$-point, then it is easy to see that Player I has a winning strategy for $G_{P\text{-point}}(U) * G_{P\text{-point}}(U)$. 
Definition
Let \( \mathcal{F} \) be a Canjar filter and \( \mathcal{W} \) a \( P \)-point. We say that \( \mathcal{F} \) is \( \mathcal{W} \)-Canjar if Player I does not have a winning strategy for \( \mathcal{G}_{\text{Canjar}}(\mathcal{F}) \ast \mathcal{G}_{\text{P-point}}(\mathcal{W}) \).

Theorem
Let \( \mathcal{F} \) be a Canjar filter and \( \mathcal{W} \) a \( P \)-point. If \( \mathcal{F} \) is \( \mathcal{W} \)-Canjar, then there is a proper forcing \( \mathbb{P}_T(\mathcal{F}) \) that diagonalizes \( \mathcal{F} \) and preserves \( \mathcal{W} \).
Theorem

Let $\mathcal{F}$ be a Canjar filter and $\mathcal{W}$ a P-point. If $\mathcal{F}$ is $\mathcal{W}$-Canjar, then there is a proper forcing $\mathbb{PT}(\mathcal{F})$ that diagonalizes $\mathcal{F}$ and preserves $\mathcal{W}$.

Well... this is not entirely correct, the correct definition of $\mathcal{W}$-Canjar is slightly more complicated, but in the same spirit (only a bit more complicated) as the one presented in the slides.
Theorem

There is a $\sigma$-closed forcing $P$ that adds a Canjar ultrafilter $U$ that is $\mathcal{W}$-Canjar for every ground model $P$-point $\mathcal{W}$.

Iterating $P * PT(U)$ will produce a model of $\omega_1 = u < s$. 
Theorem

Let $\mathcal{A}$ be a MAD family. There is a $\sigma$-closed forcing $\mathbb{P}_\mathcal{A}$ that adds a Canjar ultrafilter $\mathcal{U}_\mathcal{A}$ disjoint from $\mathcal{A}$ that is $\mathcal{W}$-Canjar for every ground model $P$-point $\mathcal{W}$.

Iterating forcings of the type $\mathbb{P}_\mathcal{A} \ast \mathbb{P}_T (\mathcal{U}_\mathcal{A})$ will produce a model of $\omega_1 = u < a = s$. 

Thank you for your attention!
Let $p \subseteq \omega^\omega$ be a tree. If $s \in p$, define $\text{suc}_p(s) = \{ n \mid s \upharpoonright n \in p \}$ . In this talk, we will say that $s \in p$ is a splitting node if $\text{suc}_p(s)$ is infinite.

**Definition**

We say that a tree $p \subseteq \omega^\omega$ is a **Miller tree** ($p \in \mathbb{P}^\mathbb{T}$) if the following conditions hold:

1. $p$ consists of increasing sequences.
2. $p$ has a stem ($t$ is the stem of $p$ if every node of $p$ is compatible with $t$ and $t$ is maximal with this property).
3. For every $s \in p$, there is $t \in p$ such that $s \subseteq t$ and $t$ is a splitting node.
If $X \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$, then $X \in (\mathcal{F}^{<\omega})^+$ if and only if for every $A \in \mathcal{F}$, there is $s \in X$ such that $s \subseteq A$.

By $\text{split} (p)$ we denote the collection of all splitting nodes and by $\text{split}_n (p)$ we denote the collection of $n$-splitting nodes (i.e. $s \in \text{split}_n (p)$ if $s \in \text{split} (p)$ and $s$ has exactly $n$-restrictions that are splitting nodes).

Given $p \in \Pi \mathbb{T}$ for every $s \in \text{split}_n (p)$ we define $F (p, s) = \{ t \setminus s \mid t \in \text{split}_{n+1} (p) \land s \subseteq t \}$.

**Definition**

Let $\mathcal{F}$ be a filter. We say $p \in \Pi \mathbb{T} (\mathcal{F})$ if the following holds:

1. $p \in \Pi \mathbb{T}$.
2. If $s \in \text{split} (p)$ then $F (p, s) \in (\mathcal{F}^{<\omega})^+$.

We order $\Pi \mathbb{T} (\mathcal{F})$ by inclusion.
Definition

Let $\mathcal{I}$ be an ideal on $\omega$. We define $\mathbb{F}_\sigma(\mathcal{I})$ as the collection of all $F_\sigma$-filters $\mathcal{F}$ such that $\mathcal{F} \cap \mathcal{I} = \emptyset$. We order $\mathbb{F}_\sigma(\mathcal{I})$ by inclusion.

Lemma

Let $\mathcal{I}$ be an ideal on $\omega$.

1. $\mathbb{F}_\sigma(\mathcal{I})$ is a $\sigma$-closed forcing.
2. $\mathbb{F}_\sigma(\mathcal{I})$ adds an ultrafilter (which we will denote by $\mathcal{U}_{\text{gen}}(\mathcal{I})$) disjoint from $\mathcal{I}$.
3. $\mathbb{F}_\sigma(\mathcal{I}) * \text{PT}(\dot{\mathcal{U}}_{\text{gen}}(\mathcal{I}))$ and $\mathbb{F}_\sigma(\mathcal{I}) * \text{M}(\dot{\mathcal{U}}_{\text{gen}}(\mathcal{I}))$ are proper forcings that destroy $\mathcal{I}$.

If $\mathcal{A}$ is a MAD family, we will denote $\mathbb{F}_\sigma(\mathcal{A})$ instead of $\mathbb{F}_\sigma(\mathcal{I}(\mathcal{A}))$ and $\mathcal{U}_{\text{gen}}(\mathcal{A})$ instead of $\mathcal{U}_{\text{gen}}(\mathcal{I}(\mathcal{A}))$. Note that $\mathbb{F}_\sigma([\omega]^{<\omega})$ is the collection of all $F_\sigma$-filters. In this case, we will only denote it by $\mathbb{F}_\sigma$ and by $\mathcal{U}_{\text{gen}}$ we will denote the generic ultrafilter added by $\mathbb{F}_\sigma$. 

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Theorem

Let $\mathcal{W}$ be a P-point and $\mathcal{A}$ a MAD family.

1. If $\mathcal{F}$ is an $F_\sigma$-filter, then $\text{PT}(\mathcal{F})$ preserves $\mathcal{W}$.
2. $F_\sigma$ forces that $\text{PT}(\dot{\mathcal{U}}_{\text{gen}})$ preserves $\mathcal{W}$.
3. $F_\sigma(\mathcal{A})$ forces that $\text{PT}(\dot{\mathcal{U}}_{\text{gen}}(\mathcal{A}))$ preserves $\mathcal{W}$. 