The ultrafilter and almost disjointness numbers

Osvaldo Guzmán joint work with Damjan Kalajdzievski

University of Toronto

August 15, 2019

The *cardinal invariants of the continuum* are uncountable cardinals whose size is at most the cardinality of the real numbers. We are mostly interested in cardinals with a nice topological or combinatorial definition.

1. By ω we denote the set (cardinal) of the natural numbers.
2. By $\mathfrak{c}$ we denote the cardinality of the real numbers.

1. The cardinal invariants of the continuum are cardinals $\mathfrak{j}$ such that:

$$\omega < \mathfrak{j} \leq \mathfrak{c}$$

2. The *Continuum Hypothesis* (*CH*) is the following statement:

$\mathfrak{c}$ is the first uncountable cardinal

3. All cardinal invariants are $\mathfrak{c}$ under *CH.*
4. *Martin's Axiom (MA)* implies that most cardinal invariants are $\mathfrak{c}$.

We are interested in studying the relationships between different cardinal invariants.

$\mathfrak{a}$	almost disjointness number
$\mathfrak{b}$	bounding number
$\mathfrak{c}$	cardinality of the continuum
$\mathfrak{d}$	dominating number
$\mathfrak{e}$	evasion number
$\mathfrak{f}$	free sequence number
$\mathfrak{g}$	groupwise number
$\mathfrak{h}$	distributivity number
$\mathfrak{i}$	independence number
$\mathfrak{j}$	
$\mathfrak{k}$	
$\mathfrak{l}$	Laver property number
$\mathfrak{m}$	Martin's number
$\mathfrak{n}$	Novak's number (might be bigger than $\mathfrak{c}$)
$\mathfrak{o}$	the offbranch number

$\mathfrak{p}$	pseudointersection number
$\mathfrak{q}$	Q-set number
$\mathfrak{r}$	reaping number
$\mathfrak{s}$	splitting number
$\mathfrak{p}$	tower number
$\mathfrak{u}$	ultrafilter number
$\mathfrak{v}$	
$\mathfrak{w}$	
$\mathfrak{x}$	
$\mathfrak{y}$	
$\mathfrak{z}$	sequence number

$$
\begin{array}{lll}
\mathfrak{hm} & \mathfrak{rr} & \mathfrak{a}_e \\
\mathfrak{hom} & \mathfrak{ss} & \mathfrak{a}_g \\
\mathfrak{sep} & \mathfrak{a}_T & \\
\mathfrak{par} & \mathfrak{ra} &
\end{array}
$$

Definition

An infinite family $\mathcal{A} \subseteq [\omega]^{\omega}$ is *almost disjoint (AD)* if the intersection of any two different elements of $\mathcal{A}$ is finite. A MAD *family* is a maximal almost disjoint family.

The *almost disjointness number* $\mathfrak{a}$ is the smallest size of a MAD family.

We say that a family $\mathcal{F} \subseteq \wp(\omega)$ is a *filter*[1] if the following conditions hold:

1. $\omega \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$.
2. If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$.
3. If $A \in \mathcal{F}$ and $A \subseteq B$ then $B \in \mathcal{F}$.
4. $\mathcal{F} \cap [\omega]^{<\omega} = \emptyset$.

The concept of a filter formalizes a kind of "largeness" notion, the elements which belong to the filter are regarded as large, while its complements are regarded as small. An *ultrafilter* is a maximal filter.

[1] By ω se denote the set of natural numbers.

1. In the same way as with MAD families, we could define an invariant as “the smallest size of an ultrafilter” but this invariant will be $\mathfrak{c}$.
2. We need the notion of an *ultrafilter base.*

Definition

We say that $\mathcal{B} \subseteq [\omega]^\omega$ is an *ultrafilter base* if the set $\{A \mid \exists B \in \mathcal{B}\,(B \subseteq A)\}$ is an ultrafilter.

1. The *ultrafilter number* $\mathfrak{u}$ denotes the smallest size of a base for an ultrafilter on ω.

$\mathfrak{a}$	the smallest size of a MAD family
$\mathfrak{u}$	the smallest size of a base for an ultrafilter on ω.

1. $\mathfrak{a}$ and $\mathfrak{u}$ are cardinal invariants.
2. $\omega \leq \mathfrak{a}, \mathfrak{u} \leq \mathfrak{c}$.
3. What is the relationship between them?

History

1. If we assume the Continuum Hypothesis, then $\omega_1 = \mathfrak{a} = \mathfrak{u} = \mathfrak{c}$.

History

1. If we assume the Continuum Hypothesis, then $\omega_1 = \mathfrak{a} = \mathfrak{u} = \mathfrak{c}$.
2. The consistency of the inequality $\mathfrak{a} < \mathfrak{u}$ is well known and easy to prove, in fact, it holds in the Cohen, random and Silver models, among many others.

History

1. If we assume the Continuum Hypothesis, then $\omega_1 = \mathfrak{a} = \mathfrak{u} = \mathfrak{c}$.
2. The consistency of the inequality $\mathfrak{a} < \mathfrak{u}$ is well known and easy to prove, in fact, it holds in the Cohen, random and Silver models, among many others.
3. Proving the consistency of the inequality $\mathfrak{u} < \mathfrak{a}$ is much harder and used to be an open problem for a long time. In fact, it follows by the theorems of Hrušák, Moore and Džamonja that the inequality $\mathfrak{u} < \mathfrak{a}$ can not be obtained by using countable support iteration of proper Borel partial orders.

The consistency of $\mathfrak{u} < \mathfrak{a}$ was finally established by Shelah, when he proved the following theorem:

Theorem (Shelah)

Let V be a model of GCH, κ a measurable cardinal and μ, λ two regular cardinals such that $\kappa < \mu < \lambda$. There is a c.c.c. forcing extension of V that satisfies $\mu = \mathfrak{b} = \mathfrak{d} = \mathfrak{u}$ and $\lambda = \mathfrak{a} = \mathfrak{c}$. In particular, CON(ZFC + "there is a measurable cardinal") implies CON(ZFC + "$\mathfrak{u} < \mathfrak{a}$").

Theorem (Shelah)

Let V be a model of GCH, κ a measurable cardinal. There is a c.c.c. forcing extension of V that satisfies $\mathfrak{u} = \kappa^+$ and $\mathfrak{a} = \mathfrak{c} = \kappa^{++}$. In particular, CON(ZFC + "there is a measurable cardinal") implies CON(ZFC + "$\mathfrak{u} < \mathfrak{a}$").

This theorem was one of the first results proved using "template iterations", which is a very powerful method that has been very useful and has been successfully applied to this day. In spite of the beauty of this result, it leaves open the following questions:

Problem (Shelah)

Does CON(ZFC) imply CON(ZFC + "$\mathfrak{u} < \mathfrak{a}$")?

Problem (Brendle)

Is it consistent that $\omega_1 = \mathfrak{u} < \mathfrak{a}$?

With Damjan Kalajdzievski, we were able provide a positive answer to both questions, by proving (without appealing to large cardinals) that every MAD family can be destroyed by a proper forcing that preserves P-points.

The method of forcing consists of adding a new set to the universe, in a similar way as adding a new root to a field. Forcing extensions are performed using partial orders.

In our case, we want to add a new set that destroys the maximality of a given MAD family, while preserving an ultrafilter base (of a P-point).

Definition

Let $\mathbb{P}$ be a partial order, $\mathcal{F}$ a filter and $\mathcal{U}$ an ultrafilter.

1. $\mathbb{P}$ *diagonalizes* $\mathcal{F}$ if $\mathbb{P}$ adds an infinite set almost contained in every element of $\mathcal{F}$.
2. $\mathbb{P}$ *preserves* $\mathcal{U}$ if $\mathcal{U}$ is the base of an ultrafilter after forcing with $\mathbb{P}$.

There are two usual forcings for diagonalizing a filter.

Definition

The *Laver forcing* $\mathbb{L}(\mathcal{F})$ *with respect to* $\mathcal{F}$ is the set of all trees p such that $suc_p(s) \in \mathcal{F}$ for every $s \in p$ extending the stem of p (where $suc_p(s) = \{n \mid s^\frown n \in p\}$). We say $p \leq q$ if $p \subseteq q$.

Definition

If $\mathcal{F}$ is a filter on ω (or on any countable set) we define the *Mathias forcing* $\mathbb{M}(\mathcal{F})$ *with respect to* $\mathcal{F}$ as the set of all pairs (s, A) where $s \in [\omega]^{<\omega}$ and $A \in \mathcal{F}$. If $(s, A), (t, B) \in \mathbb{M}(\mathcal{F})$ then $(s, A) \leq (t, B)$ if the following conditions hold:

1. t is an initial segment of s.
2. $A \subseteq B$.
3. $(s \setminus t) \subseteq B$.

1. Let $f, g \in \omega^\omega$, define $f \leq^* g$ if and only if $f(n) \leq g(n)$ holds for all $n \in \omega$ except finitely many. We say a family $\mathcal{B} \subseteq \omega^\omega$ is *unbounded* if $\mathcal{B}$ is unbounded with respect to $\leq^*$.
2. The *bounding number* $\mathfrak{b}$ is the size of the smallest unbounded family.
3. We say that S *splits* X if $S \cap X$ and $X \setminus S$ are both infinite. A family $\mathcal{S} \subseteq [\omega]^\omega$ is a *splitting family* if for every $X \in [\omega]^\omega$ there is $S \in \mathcal{S}$ such that S splits X.
4. The *splitting number* $\mathfrak{s}$ is the smallest size of a splitting family.

It is not difficult to prove that $\mathfrak{b} \leq \mathfrak{a}$ and $\mathfrak{b} \leq \mathfrak{u}$.

Our model will be a model of $\omega_1 = \mathfrak{b} = \mathfrak{u} < \mathfrak{a} = \mathfrak{s} = \omega_2$. We will first explain how to build a model of $\mathfrak{u} < \mathfrak{s}$.

Theorem (Blass-Shelah)

The inequality $\mathfrak{u} < \mathfrak{s}$ is consistent.

It is easy to see that diagonalizing an ultrafilter destroys all ground model splitting families. In this way, if we want to build a model of $\mathfrak{u} < \mathfrak{s}$, we need to diagonalize an ultrafilter, while preserving another one (in fact, preserving a P-point). This topic has also been recently studied by Heike Mildenberger.

While $\mathbb{L}(\mathcal{F})$ always adds a dominating real, this may not be the case for $\mathbb{M}(\mathcal{F})$. A trivial example is taking $\mathcal{F}$ to be the cofinite filter in ω, since in this case $\mathbb{M}(\mathcal{F})$ is forcing equivalent to Cohen forcing. A more interesting example was found by Canjar, where an ultrafilter whose Mathias forcing does not add dominating reals was constructed under $\mathfrak{d} = \mathfrak{c}$.

Definition

We say that a filter $\mathcal{F}$ is *Canjar* if $\mathbb{M}(\mathcal{F})$ does not add dominating reals.

In order to provide a combinatorial characterization of the previous notion, we need the following definition:

Definition

Let $\mathcal{F}$ be a filter on ω. Define the filter $\mathcal{F}^{<\omega}$ in $[\omega]^{<\omega} \setminus \{\emptyset\}$ as the filter generated by $\left\{ [A]^{<\omega} \setminus \{\emptyset\} \mid A \in \mathcal{F} \right\}$.

Note that if $X \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$, then $X \in (\mathcal{F}^{<\omega})^{+}$ if and only if for every $A \in \mathcal{F}$, there is $s \in X$ such that $s \subseteq A$.

Theorem

Let $\mathcal{F}$ be a filter on ω. The following are equivalent:

1. $\mathcal{F}$ *is Canjar.*
2. *(Hrušák, Minami) For every* $\{X_n \mid n \in \omega\} \subseteq (\mathcal{F}^{<\omega})^+$ *there are* $Y_n \in [X_n]^{<\omega}$ *such that* $\bigcup_{n\in\omega} Y_n \in (\mathcal{F}^{<\omega})^+$.
3. *(Chodounský, Repovš and Zdomskyy)* $\mathcal{F}$ *is Menger (as a subspace of* $\wp(\omega) \simeq 2^\omega$*).*[a]

[a] We view filters as subspaces of 2^ω, the notion of Borel or F_σ is taken using the usual topology on 2^ω.

Let $\mathcal{F}$ be a filter. The *Canjar game* $\mathcal{G}_{Canjar}(\mathcal{F})$ is defined as follows:

I	X_0		X_1		X_2		...
II		Y_0		Y_1		Y_2	

Where $X_i \in (\mathcal{F}^{<\omega})^+$ and $Y_i \in [X_i]^{<\omega}$ for every $i \in \omega$. The player II *wins the game* $\mathcal{G}_{Canjar}(\mathcal{F})$ if $\bigcup_{n\in\omega} Y_n \in (\mathcal{F}^{<\omega})^+$.

Theorem (Chodounský, Repovš and Zdomskyy)

Let $\mathcal{F}$ be a filter on ω. The following are equivalent:

1. *$\mathcal{F}$ is Canjar.*
2. *Player* I *does not have a winning strategy in $\mathcal{G}_{Canjar}(\mathcal{F})$.*

Definition

$\mathcal{U}$ is a *P-point* if every countable subfamily $\mathcal{B} \subseteq \mathcal{U}$ there is $A \in \mathcal{U}$ such that $A \setminus B$ is finite for every $B \in \mathcal{B}$.

Let $\mathcal{U}$ be an ultrafilter. Recall that the *P-point game* $\mathcal{G}_{P\text{-}point}(\mathcal{U})$ is defined as follows:

I	W_0		W_1		...
II		z_0		z_1	

Where $W_i \in \mathcal{U}$ and $z_i \in [W_i]^{<\omega}$. The player II will *win the game* $\mathcal{G}_{P\text{-}point}(\mathcal{U})$ if $\bigcup_{m \in \omega} z_m \in \mathcal{U}$. It is well known that player II can not have a winning strategy for this game. The following is a well known result of Galvin and Shelah:

Theorem (Galvin-Shelah)

Let $\mathcal{U}$ be an ultrafilter. $\mathcal{U}$ is a P-point if and only if Player I *does not have a winning strategy in $\mathcal{G}_{P\text{-}point}(\mathcal{U})$.*

Let $\mathcal{G}$ and $\mathcal{H}$ be two (infinite) games:

$\mathcal{G}:$	I	a_0		a_1		...
	II		b_0		b_1	

$\mathcal{H}:$	I	c_0		c_1		...
	II		d_0		d_1	

We define the game $\mathcal{G} * \mathcal{H}$ as follows:

$\mathcal{G} * \mathcal{H}:$	I	a_0		c_0		a_1		c_1		...
	II		b_0		d_0		b_1		d_1	

Where $\langle a_i, b_i \rangle_{i \in \omega}$ is played according to $\mathcal{G}$ and $\langle c_i, d_i \rangle_{i \in \omega}$ is played according to $\mathcal{H}$. Player II will win $\mathcal{G} * \mathcal{H}$ is $\langle a_i, b_i \rangle_{i \in \omega}$ is a winning run for Player II in $\mathcal{G}$ and $\langle c_i, d_i \rangle_{i \in \omega}$ is a winning run for Player II in $\mathcal{H}$.

Let $\mathcal{G}$ and $\mathcal{H}$ be two games. It seems obvious that if Player I does not have a winning strategy for $\mathcal{G}$ or $\mathcal{H}$, then he will not have a winning strategy for $\mathcal{G} * \mathcal{H}$ ... but this is false.

If $\mathcal{U}$ is a P-point, then it is easy to see that Player I has a winning strategy for $\mathcal{G}_{P\text{-}point}(\mathcal{U}) * \mathcal{G}_{P\text{-}point}(\mathcal{U})$.

Definition

Let $\mathcal{F}$ be a Canjar filter and $\mathcal{W}$ a P-point. We say that $\mathcal{F}$ is *$\mathcal{W}$-Canjar* if Player I does not have a winning strategy for $\mathcal{G}_{Canjar}(\mathcal{F}) * \mathcal{G}_{P\text{-}point}(\mathcal{W})$.

Theorem

Let $\mathcal{F}$ be a Canjar filter and $\mathcal{W}$ a P-point. If $\mathcal{F}$ is $\mathcal{W}$-Canjar, then there is a proper forcing $\mathbb{PT}(\mathcal{F})$ that diagonalizes $\mathcal{F}$ and preserves $\mathcal{W}$.

Theorem

Let $\mathcal{F}$ be a Canjar filter and $\mathcal{W}$ a P-point. If $\mathcal{F}$ is $\mathcal{W}$-Canjar, then there is a proper forcing $\mathbb{PT}\left(\mathcal{F}\right)$ that diagonalizes $\mathcal{F}$ and preserves $\mathcal{W}$.

Well... this is not entirely correct, the correct definition of $\mathcal{W}$-Canjar is slightly more complicated, but in the same spirit (only a bit more complicated) as the one presented in the slides.

Theorem

There is a σ-closed forcing $\mathbb{P}$ that adds a Canjar ultrafilter $\mathcal{U}$ that is $\mathcal{W}$-Canjar for every ground model P-point $\mathcal{W}$.

Iterating $\mathbb{P} * \mathbb{PT}(\mathcal{U})$ will produce a model of $\omega_1 = \mathfrak{u} < \mathfrak{s}$.

Theorem

Let $\mathcal{A}$ be a MAD family. There is a σ-closed forcing $\mathbb{P}_{\mathcal{A}}$ that adds a Canjar ultrafilter $\mathcal{U}_{\mathcal{A}}$ disjoint from $\mathcal{A}$ that is $\mathcal{W}$-Canjar for every ground model P-point $\mathcal{W}$.

Iterating forcings of the type $\mathbb{P}_{\mathcal{A}} * \mathbb{PT}(\mathcal{U}_{\mathcal{A}})$ will produce a model of $\omega_1 = \mathfrak{u} < \mathfrak{a} = \mathfrak{s}$.

Thank you for your attention!

Let $p \subseteq \omega^{<\omega}$ be a tree. If $s \in p$, define $suc_p(s) = \{n \mid s^\frown n \in p\}$. In this talk, we will say that $s \in p$ is a *splitting node* if $suc_p(s)$ is **infinite.**

Definition

We say that a tree $p \subseteq \omega^{<\omega}$ is a *Miller tree* ($p \in \mathbb{PT}$) if the following conditions hold:

1. p consists of increasing sequences.
2. p has a stem (t is the stem of p if every node of p is compatible with t and t is maximal with this property).
3. For every $s \in p$, there is $t \in p$ such that $s \subseteq t$ and t is a splitting node.

If $X \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$, then $X \in (\mathcal{F}^{<\omega})^{+}$ if and only if for every $A \in \mathcal{F}$, there is $s \in X$ such that $s \subseteq A$.

By $split(p)$ we denote the collection of all splitting nodes and by $split_n(p)$ we denote the collection of n-splitting nodes (i.e. $s \in split_n(p)$ if $s \in split(p)$ and s has exactly n-restrictions that are splitting nodes). Given $p \in \mathbb{PT}$ for every $s \in split_n(p)$ we define $F(p, s) = \{t \setminus s \mid t \in split_{n+1}(p) \wedge s \subseteq t\}$.

Definition

Let $\mathcal{F}$ be a filter. We say $p \in \mathbb{PT}(\mathcal{F})$ if the following holds:

1. $p \in \mathbb{PT}$.
2. If $s \in split(p)$ then $F(p, s) \in (\mathcal{F}^{<\omega})^{+}$.

We order $\mathbb{PT}(\mathcal{F})$ by inclusion.

Definition

Let $\mathcal{I}$ be an ideal on ω. We define $\mathbb{F}_\sigma(\mathcal{I})$ as the collection of all F_σ-filters $\mathcal{F}$ such that $\mathcal{F} \cap \mathcal{I} = \emptyset$. We order $\mathbb{F}_\sigma(\mathcal{I})$ by inclusion.

Lemma

Let $\mathcal{I}$ be an ideal on ω.

1. $\mathbb{F}_\sigma(\mathcal{I})$ *is a σ-closed forcing.*
2. $\mathbb{F}_\sigma(\mathcal{I})$ *adds an ultrafilter (which we will denote by $\mathcal{U}_{gen}(\mathcal{I})$) disjoint from $\mathcal{I}$.*
3. $\mathbb{F}_\sigma(\mathcal{I}) * \mathbb{PT}(\dot{\mathcal{U}}_{gen}(\mathcal{I}))$ *and* $\mathbb{F}_\sigma(\mathcal{I}) * \mathbb{M}(\dot{\mathcal{U}}_{gen}(\mathcal{I}))$ *are proper forcings that destroy $\mathcal{I}$.*

If $\mathcal{A}$ is a MAD family, we will denote $\mathbb{F}_\sigma(\mathcal{A})$ instead of $\mathbb{F}_\sigma(\mathcal{I}(\mathcal{A}))$ and $\mathcal{U}_{gen}(\mathcal{A})$ instead of $\mathcal{U}_{gen}(\mathcal{I}(\mathcal{A}))$. Note that $\mathbb{F}_\sigma\left([\omega]^{<\omega}\right)$ is the collection of all F_σ-filters. In this case, we will only denote it by $\mathbb{F}_\sigma$ and by $\mathcal{U}_{gen}$ we will denote the generic ultrafilter added by $\mathbb{F}_\sigma$.

Theorem

Let $\mathcal{W}$ be a P-point and $\mathcal{A}$ a MAD family.

1. *If $\mathcal{F}$ is an F_σ-filter, then $\mathbb{PT}(\mathcal{F})$ preserves $\mathcal{W}$.*
2. *$\mathbb{F}_\sigma$ forces that $\mathbb{PT}(\dot{\mathcal{U}}_{gen})$ preserves $\mathcal{W}$.*
3. *$\mathbb{F}_\sigma(\mathcal{A})$ forces that $\mathbb{PT}(\dot{\mathcal{U}}_{gen}(\mathcal{A}))$ preserves $\mathcal{W}$.*