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We introduce a weak set theory $\mathsf{H}_{<\omega}$. A formalization of arithmetic on finite von Neumann ordinals gives an embedding of arithmetical language into this theory. We show that $\mathsf{H}_{<\omega}$ proves a natural arithmetization of its own Hilbert-style consistency. Unlike the previous examples (due to Willard [2]) of theories proving their own consistency, $\mathsf{H}_{<\omega}$ appears to be sufficiently natural.

The theory $\mathsf{H}_{<\omega}$ is infinitely axiomatizable and proves existence of all individual hereditarily finite sets, but at the same time all its finite subtheories have finite models. Therefore, our example avoids the strong version of Gödel's second incompleteness theorem (due to Pudlák) that asserts that no consistent theory interpreting Robinson's arithmetic Q proves its own consistency [1]. To show that $\mathsf{H}_{<\omega}$ proves its own consistency we establish a conservation result connecting Kalmar elementary arithmetic EA and $\mathsf{H}_{<\omega}$.

The theory $\mathsf{H}_{<\omega}$ is first-order theory in the signature with equality $=$, membership predicate $\in$, and unary function $\overline{\mathsf{V}}$. Axioms of $\mathsf{H}_{<\omega}$:

1. $x = y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y)$ (Extensionality);
2. $\exists y \forall z(z \in y \leftrightarrow z \in x \wedge \varphi(z))$ (Separation);
3. $y \in \overline{\mathsf{V}}(x) \leftrightarrow (\exists z \in x)(y \subseteq \overline{\mathsf{V}}(z))$ (Defining axiom for $\overline{\mathsf{V}}$);
4. $\exists x\, \mathsf{Nat}_n(x)$, for all $n \in \mathbb{N}$ (all individual natural numbers exist).

Here the formulas $\mathsf{Nat}_n(x)$ expressing the fact that x is the ordinal n are defined in the usual manner: $\mathsf{Nat}_0(x)$ is $\forall y\ y \notin x$ and $\mathsf{Nat}_{n+1}(x)$ is $\forall y\ (\mathsf{Nat}_n(y) \to \forall z(z \in x \leftrightarrow z = y \vee z \in y))$. The intended interpretation of the function $\overline{\mathsf{V}}$ is $\overline{\mathsf{V}}\colon x \longmapsto \mathsf{V}_\alpha$, where α is least ordinal such that $x \subseteq \mathsf{V}_\alpha$.

[1] P. Pudlk, *Provability algebras and proof-theoretic ordinals*, ***The Journal of Symbolic Logic***, vol. 50 (1985), nmb. 2, pp. 423–441.

[2] D.E. Willard, *A generalization of the Second Incompleteness Theorem and some exceptions to it*, ***Annals of Pure and Applied Logic***, vol. 141 (2006), nmb. 3, pp. 472–496.