▶ TAISHI KURAHASHI, *Derivability conditions and the second incompleteness theorem.*
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Let T be any recursively axiomatized consistent extension of Peano arithmetic. In his famous paper, Gödel showed that the consistency statement $\text{Con}_T \equiv \exists x(\mathsf{Fml}(x) \wedge \text{Pr}_T(x))$ cannot be proved in T. In the second volume of *Grundlagen der Mathematik*, Hilbert and Bernays proposed a set of conditions for provability predicates which is sufficient for a version of the second incompleteness theorem. That is, if $\text{Pr}_T(x)$ is a Σ_1 provability predicate satisfying their conditions, then $\text{Con}_T^0 \equiv \forall x(\mathsf{Fml}(x) \wedge \text{Pr}_T(x) \to \neg\text{Pr}_T(\neg x))$ cannot be proved in T. Löb [4] found another set of conditions, and proved the so-called Löb's theorem under his conditions. Löb's theorem immediately implies that $\text{Con}_T^1 \equiv \text{Pr}_T(\ulcorner 0 \neq 0 \urcorner)$ cannot be proved in T. Notice that for provability predicates, Con_T^0 implies Con_T^1, and Con_T^1 implies Con_T.

Related to derivability conditions and the second incompleteness theorem, we proved the following results.

1. There are new sets of derivability conditions which are sufficient for unprovability of Con_T^0.
2. If a Σ_1 provability predicate $\text{Pr}_T(x)$ satisfies the following condition $\mathbf{B_2^U}$, then $\text{Pr}_T(x)$ satisfies provable Σ_1-completeness.

   $\mathbf{B_2^U}$: If $T \vdash \varphi(\vec{x}) \to \psi(\vec{x})$, then $T \vdash \text{Pr}_T(\ulcorner\varphi(\dot{\vec{x}})\urcorner) \to \text{Pr}_T(\ulcorner\psi(\dot{\vec{x}})\urcorner)$

   This is an improvement of Buchholz's observation [1].
3. Hilbert and Bernays's conditions and Löb's conditions are mutually incomparable.
4. Both of Hilbert and Bernays' conditions and the global versions of Löb's conditions are not sufficient for $T \nvdash \text{Con}_T$. This shows that both of Hilbert-Bernays' conditions and Löb's conditions do not accomplish Gödel's original statement of the second incompleteness theorem.

[1] WILFRIED BUCHHOLZ, *Mathematische Logik II*, http://www.mathematik.uni-muenchen.de/~buchholz/articles/LogikII.ps (1993).

[2] TAISHI KURAHASHI, *A note on derivability conditions*, arXiv:1902.00895

[3] TAISHI KURAHASHI, *Rosser provability and the second incompleteness theorem*, arXiv:1902.06863

[4] MARTIN HUGO LÖB, *Solution of a problem of Leon Henkin*, **The Journal of Symbolic Logic**, vol. 20 (1955), no. 2, pp. 115–118.