

- FRANCO PARLAMENTO, FLAVIO PREVIALE, FEDERICO MUNINI, *The subterm property for some equality sequent calculi*.

Department of Mathematics, Computer Science and Physics, University of Udine, via Delle Scienze 206, 33100 Udine, Italy.

E-mail: franco.parlamento@uniud.it.

As for *cut elimination* (see [3] p. 93), we say that the *subterm property* holds for a sequent calculus \mathbf{S} if there is a “non-trivial” algorithm for transforming a derivation in \mathbf{S} of a sequent S into a derivation of S in the same system, that contains only terms occurring in S . We show that the subterm property holds for the following purely equality calculi based on the structural rules:

1. $\ast\mathbf{EQ}_N$ (N for “natural”), which has the reflexivity axioms $\Rightarrow t = t$ and the multiple congruence rule

$$\frac{\Gamma \Rightarrow r_1 = s_1 \quad \dots \quad \Gamma \Rightarrow r_n = s_n \quad \Gamma \Rightarrow F[v_1/r_1, \dots, v_n/r_n]}{\Gamma \Rightarrow F[v_1/s_1, \dots, v_n/s_n]}$$

2. $\ast\mathbf{EQ}_B$ (B for “Birkhoff”), which has the reflexivity axioms and the rules:

$$\frac{\Gamma \Rightarrow r = s}{\Gamma \Rightarrow s = r} \quad \frac{\Gamma \Rightarrow r = s \quad \Gamma \Rightarrow s = t}{\Gamma \Rightarrow r = t}$$

$$\frac{\Gamma \Rightarrow r_1 = s_1 \quad \dots \quad \Gamma \Rightarrow r_n = s_n}{\Gamma, P[v_1/r_1, \dots, v_n/r_n] \Rightarrow P[v_1/s_1, \dots, v_n/s_n]}$$

$$\frac{\Gamma \Rightarrow r_1 = s_1 \quad \dots \quad \Gamma \Rightarrow r_n = s_n}{\Gamma \Rightarrow t[v_1/r_1, \dots, v_n/r_n] = t[v_1/s_1, \dots, v_n/s_n]}$$

3. $\ast\mathbf{EQ}$, which has the reflexivity axioms and the rules

$$\frac{\Gamma \Rightarrow F[v_1/r_1, \dots, v_n/r_n]}{r_1 = s_1, \dots, r_n = s_n, \Gamma \Rightarrow F[v_1/s_1, \dots, v_n/s_n]}$$

$$\frac{\Gamma \Rightarrow F[v_1/r_1, \dots, v_n/r_n]}{s_1 = r_1, \dots, s_n = r_n, \Gamma \Rightarrow F[v_1/s_1, \dots, v_n/s_n]}$$

where Γ is a finite multiset of formulae, F is a formula, P is an atomic formula different from an equality, r, s, t , the r_i 's and s_i 's are terms and the v_i 's are variable of a first order language and $E[v_i/t_1, \dots, v_i/t_n]$ is used to denote the result of the simultaneous replacement of the free variables v_1, \dots, v_n by the terms t_1, \dots, t_n in the formula or term E .

Moreover, for $\ast\mathbf{EQ}_N$ and $\ast\mathbf{EQ}_B$ cut elimination and the subterm property hold simultaneously, namely a derivation in any of such systems of a sequent S can be transformed into a cut-free derivation of S in the same system, containing only terms occurring in S . Although cut elimination holds also for $\ast\mathbf{EQ}$, it does not hold simultaneously with the subterm property.

[1] F. PARLAMENTO, F. PREVIALE, *Absorbing the Structural Rules in the Sequent Calculus with Additional Atomic Rules*, *Logic Colloquium 2018 - Contributed Talk ArXiv 1810.11407*

[2] F. PARLAMENTO, F. PREVIALE, *The Cut Elimination and Nonlengthening Property for the Sequent Calculus with Equality*, *Logic Colloquium 2016 The Bulletin of Symbolic Logic*, vol. 23 (2017), no. 2, pp. 251–252. *ArXiv 1705.00693*

[3] A.S. TROELSTRA, H. SCHWICHTENBERG, *Basic Proof Theory - 2nd edition*, Cambridge University Press, Cambridge, 2000