Descending Sequences of Hyperdegrees and the Second Incompleteness Theorem

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Main theme of the talk

There is a connection between Gödel's second incompleteness theorem and well-foundedness of certain computability-theoretic partial orders

A Theorem

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There is no sequence of reals $A_0, A_1, \ldots$ such that for each n

$$\mathcal{O}^{A_{n+1}} \leq_H A_n.$$

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First proof

By results of Spector, if $\mathcal{O}^A \leq_H B$ then $\omega_1^A < \omega_1^B$. So if $A_0, A_1, A_2, \ldots$ was such a sequence then we would have

$$\omega_1^{A_0} > \omega_1^{A_1} > \omega_1^{A_2} > \ldots$$

Some Suggestive Facts

Definition

A β-model is an ω-model of second order arithmetic that is correct about all Σ^1_1 facts

Fact

(ACA_0 proves) $\mathcal{O}^X$ exists $\implies$ X is contained in a β-model

Fact

(ACA_0 proves) β-models satisfy ACA_0

Incompleteness $\implies$ Well-foundedness

An alternative proof?

- Work in the theory $T = \mathsf{ACA}_0 +$ "there is such a sequence"
- Let $A_0, A_1, \ldots$ be such a descending sequence
- $\mathcal{O}^{A_1}$ exists so there is a β-model containing A_1
- The β-model satisfies ACA_0 and contains $A_1, A_2, \ldots$

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- Let $A_0, A_1, \ldots$ be such a descending sequence
- $\mathcal{O}^{A_1}$ exists so there is a β-model containing A_1
- The β-model satisfies ACA_0 and contains $A_1, A_2, \ldots$ but not necessarily a real coding this sequence
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A Correct Proof

The main idea is to show that if there is a descending sequence then there is one which is relatively simple—e.g. hyperarithmetic in A_1.

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Kleene Basis Theorem

(ACA_0 proves) If X is a real such that $\mathcal{O}^X$ exists and if φ is a Σ^1_1 formula and there is some real Y such that $\varphi(X, Y)$ holds then there is some real Y such that $Y \leq_T \mathcal{O}^X$

So it suffices to show that there is a nonempty $\Sigma^1_1(A_2)$ class consisting only of descending sequences

Incompleteness $\implies$ Well-foundedness

Observation

This proof shows that the theorem is provable in ACA_0 which is not apparent from the first proof.

Incompleteness $\implies$ Well-foundedness

General Strategy

- Work in some theory T and assume there is a descending sequence
- Show that there is a model of T containing a tail of the sequence
- Tail of a descending sequence is still a descending sequence
- Conclude that $T +$ "there is a descending sequence" proves its own consistency

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Main Difficulty

Need to pick a theory T that is weak enough that the existence of a descending sequence guarantees models of the theory exist but strong enough to prove that the descending sequence guarantees this.

Incompleteness $\Longrightarrow$ Well-foundedness

Example 1: Hyperdegrees

There is no sequence of hyperdegrees, each hyp above the hyperjump of the next

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Theorem (Steel)

If P is an arithmetic relation then there is no sequence $A_0, A_1, A_2, \ldots$ such that for each n

- A_{n+1} is the unique X such that $P(A_n, X)$
- $A'_{n+1} \leq_T A_n$

Steel's original proof used only recursion theory, but Harvey Friedman later gave a proof along the lines of the general strategy outlined here

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Well-foundedness $\implies$ Incompleteness

The well-foundedness of computability-theoretic partial orders can sometimes imply semantic versions of the second incompleteness theorem

Semantic Versions of Second Incompleteness

Gödel's second incompleteness theorem

A consistent theory cannot prove its own consistency

Consistent = has a model

Semantic version of second incompleteness

If T has a model then it has a model with no models coded in it.

Typically T is a theory of second order arithmetic which is strong enough to prove Gödel's completeness theorem.

By changing what type of models we consider, we can get statements that do not trivially follow from the usual second incompleteness theorem.

Well-foundedness $\Longrightarrow$ Incompleteness

Theorem (Mummert–Simpson)

If T is a theory in the language of second order arithmetic such that T has a β-model, then T has a β-model that contains no countable coded β-models of T

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Proof sketch

- Suppose not. Find a sequence of β-models $M_0, M_1, M_2, \ldots$ such that each M_{n+1} is coded in M_n
- Since M_n is a β-model it is correct about all Π^1_1 facts about M_{n+1}
- So $\mathcal{O}^{M_{n+1}}$ is arithmetic in M_n
- So $\mathcal{O}^{M_{n+1}} \leq_H M_n$, contradicting the first theorem in this talk

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Main Observation

The lack of a minimal model can imply a descending sequence in some computability-theoretic partial order.

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Example 1: β-models

If T is any theory such that T has a β-model then T has a β-model with no countable coded β-models of T.

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If T is any theory such that T has a β-model then T has a β-model with no countable coded β-models of T.

Example 2: ω-models of a theory extending ACA_0 (Steel)

If T is an arithmetically axiomatized theory extending ACA_0 such that T has an ω-model then T has an ω-model with no countable coded ω-models of T.

Thank you!