Towards an independent version of Tarski's system of geometry

Pierre Boutry

University of Strasbourg - ICube - CNRS

Logic Colloquium 2019
Prague, August 15, 2019

Introduction

Introduction

- Euclid's *Elements*

Introduction

- Euclid's *Elements* led to what is likely the first independence proof.

Introduction

- Euclid's *Elements* led to what is likely the first independence proof.
- Hilbert's *Grundlagen der Geometrie*

Introduction

- Euclid's *Elements* led to what is likely the first independence proof.
- Hilbert's *Grundlagen der Geometrie* contains a chapter dedicated to independence properties.

Introduction

- Euclid's *Elements* led to what is likely the first independence proof.
- Hilbert's *Grundlagen der Geometrie* contains a chapter dedicated to independence properties.
- Tarski's System of Geometry

Introduction

- Euclid's *Elements* led to what is likely the first independence proof.
- Hilbert's *Grundlagen der Geometrie* contains a chapter dedicated to independence properties.
- Tarski's System of Geometry and the problem of its independence was carefully studied by Gupta.

Outline

1. Introduction
2. Tarski's system of geometry
3. Gupta's and Szczerba's contributions
4. An independent version of Tarski's system of geometry?
5. Conclusion

Outline

1 Introduction

2 Tarski's system of geometry

3 Gupta's and Szczerba's contributions

4 An independent version of Tarski's system of geometry?

5 Conclusion

Tarski's system of geometry

Tarski's system of geometry

- A single primitive type: point.

Alfred Tarski
(1901 - 1983)

Tarski's system of geometry

- A single primitive type: point.
- Two primitive predicates:

Alfred Tarski
(1901 - 1983)

Tarski's system of geometry

- A single primitive type: point.
- Two primitive predicates:
  1. congruence $AB \equiv CD$;

Alfred Tarski
(1901 - 1983)

Tarski's system of geometry

- A single primitive type: point.
- Two primitive predicates:
  1. congruence $AB \equiv CD$;
  2. betweenness $A—B—C$.

Alfred Tarski
(1901 - 1983)

Tarski's system of geometry

- A single primitive type: point.
- Two primitive predicates:
  1. congruence $AB \equiv CD$;
  2. betweenness $A—B—C$.
- 11 axioms.

Alfred Tarski
(1901 - 1983)

Tarski's system of geometry

- A single primitive type: point.
- Two primitive predicates:
  1. congruence $AB \equiv CD$;
  2. betweenness $A—B—C$.
- 11 axioms.
- A parameter controls the dimension.

Alfred Tarski
(1901 - 1983)

Tarski's system of geometry

- A single primitive type: point.
- Two primitive predicates:
  1. congruence $AB \equiv CD$;
  2. betweenness $A—B—C$.
- 11 axioms.
- A parameter controls the dimension.
- Good meta-theoritical properties.

Alfred Tarski
(1901 - 1983)

Axioms about congruence

Axioms about congruence

Axiom (Pseudo-transitivity for congruence)

$$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$$

Axioms about congruence

Axiom (Pseudo-transitivity for congruence)

$$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$$

Axiom (Pseudo-reflexivity for congruence)

$$AB \equiv BA$$

Axioms about congruence

Axiom (Pseudo-transitivity for congruence)

$$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$$

Axiom (Pseudo-reflexivity for congruence)

$$AB \equiv BA$$

Axiom (Identity for congruence)

$$AB \equiv CC \Rightarrow A = B$$

Axiom about betweenness

Axiom about betweenness

Axiom (Identity for betweenness)

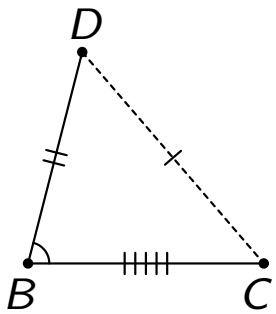
$$A\text{—}B\text{—}A \Rightarrow A = B$$

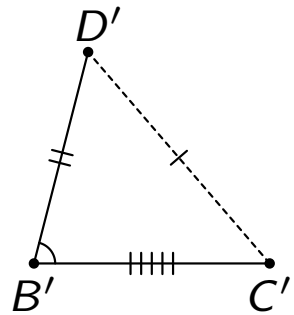
Five-Segment Axiom

Five-Segment Axiom

Axiom (Five-Segment)

$$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$$
$$AD \equiv A'D' \wedge BD \equiv B'D' \wedge$$
$$A{-}B{-}C \wedge A'{-}B'{-}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$$

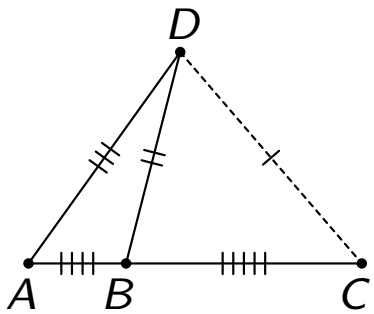


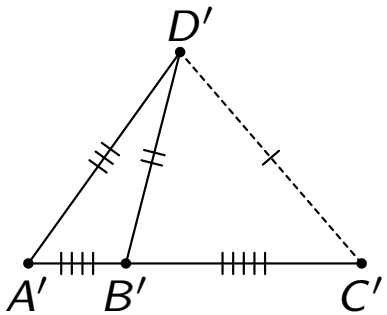


Five-Segment Axiom

Axiom (Five-Segment)

$$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$$
$$AD \equiv A'D' \wedge BD \equiv B'D' \wedge$$
$$A\text{—}B\text{—}C \wedge A'\text{—}B'\text{—}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$$



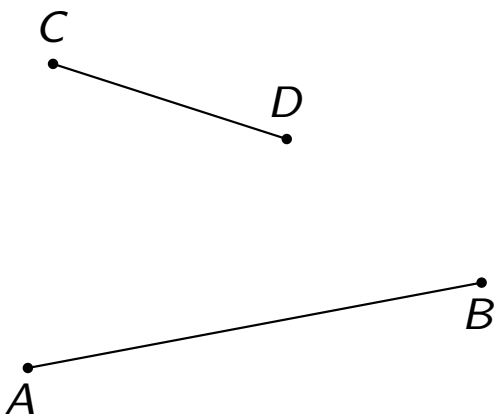


Axiom of Segment Construction

Axiom of Segment Construction

Axiom (Segment Construction)

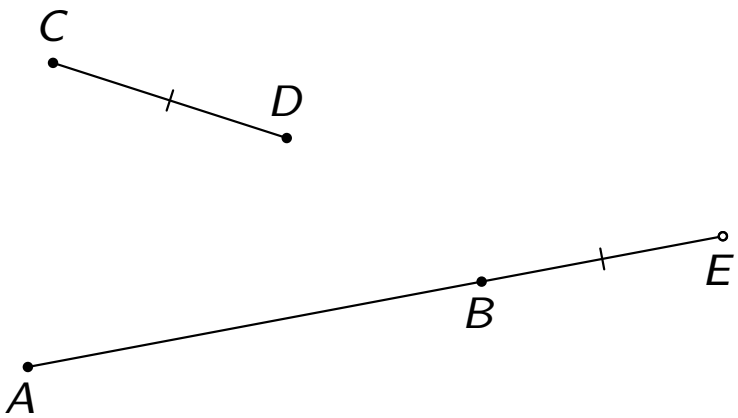
$$\exists E, A\text{—}B\text{—}E \wedge BE \equiv CD$$



Axiom of Segment Construction

Axiom (Segment Construction)

$$\exists E, A\text{—}B\text{—}E \wedge BE \equiv CD$$

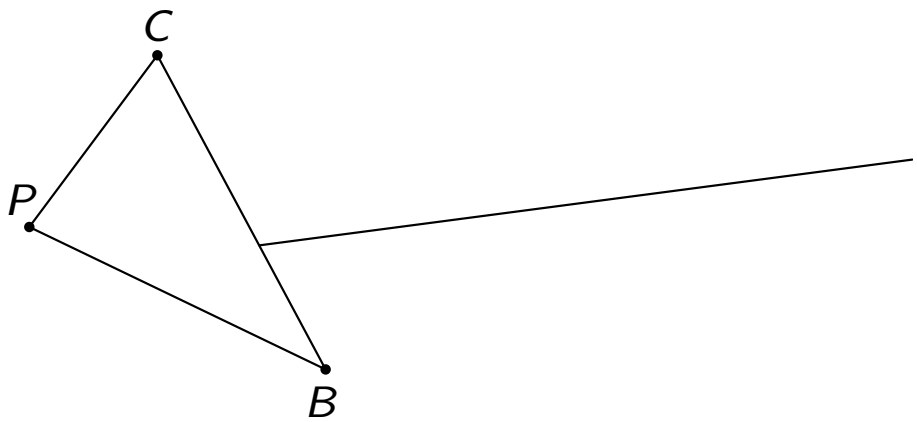


Pasch's axiom

Pasch's axiom

Axiom (Pasch)

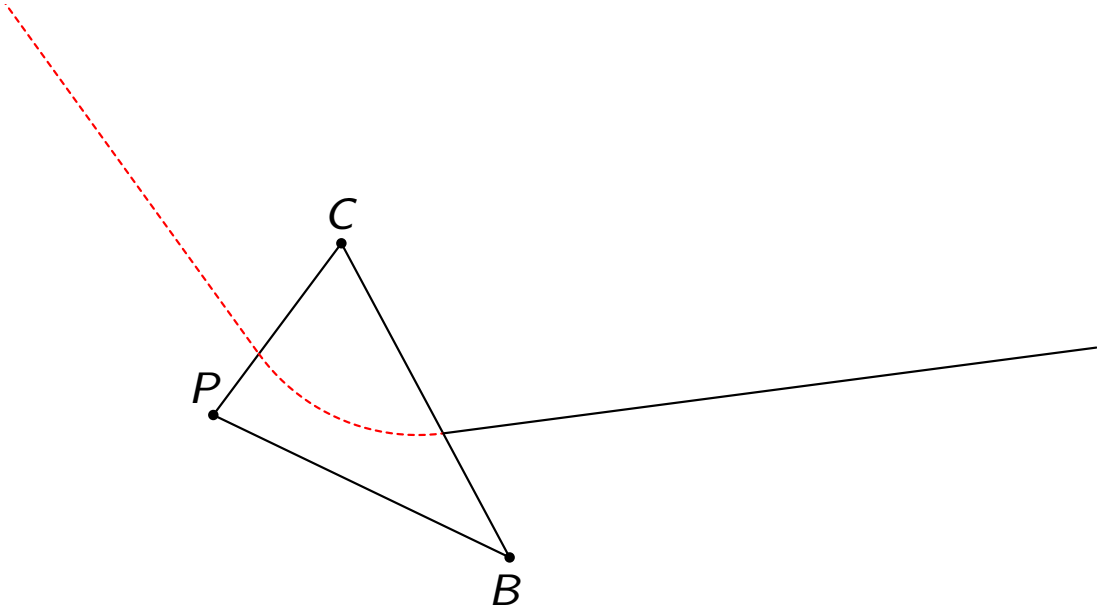
$$A\text{—}P\text{—}C \wedge B\text{—}Q\text{—}C \Rightarrow \exists X, P\text{—}X\text{—}B \wedge Q\text{—}X\text{—}A$$



Pasch's axiom

Axiom (Pasch)

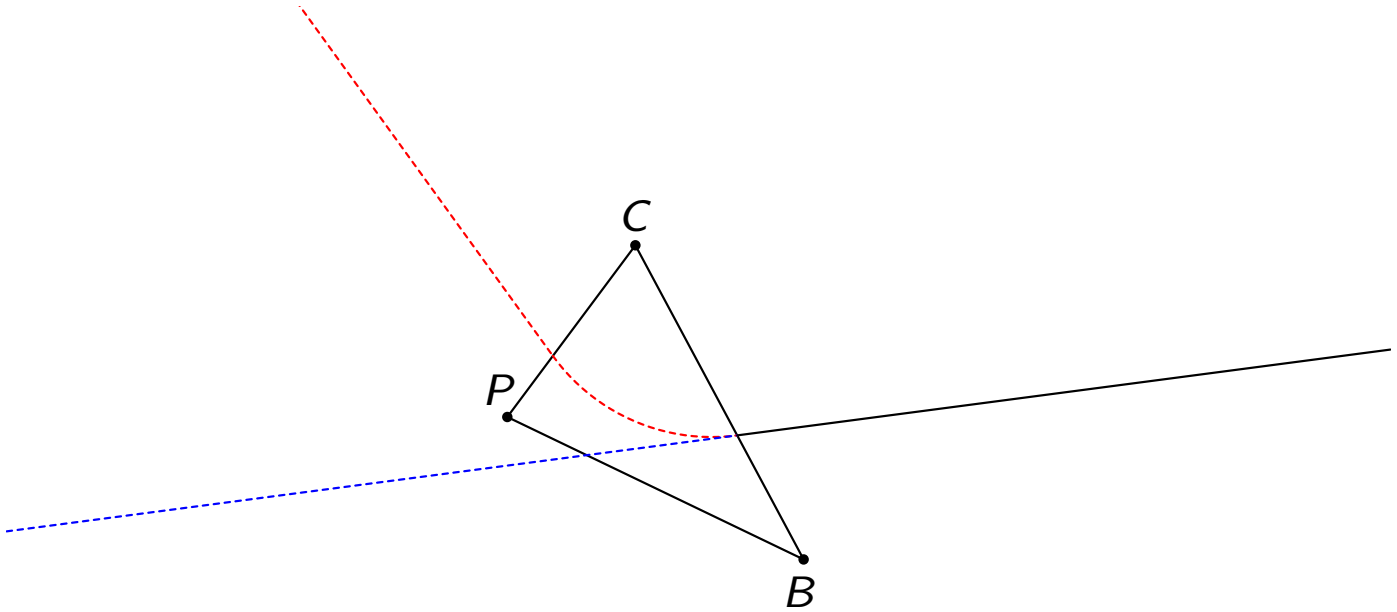
$$A\text{—}P\text{—}C \wedge B\text{—}Q\text{—}C \Rightarrow \exists X, P\text{—}X\text{—}B \wedge Q\text{—}X\text{—}A$$



Pasch's axiom

Axiom (Pasch)

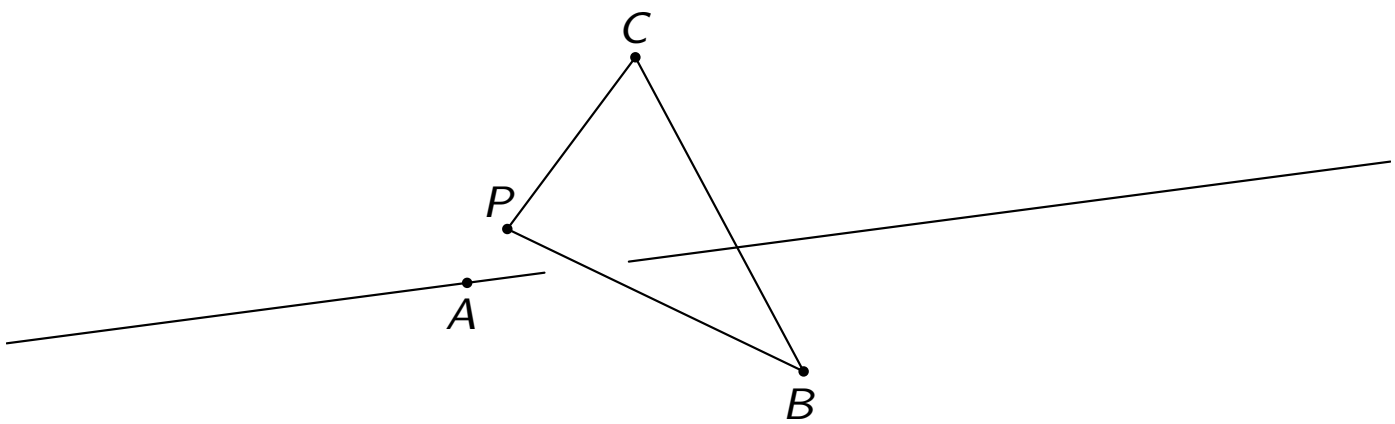
$$A\text{—}P\text{—}C \wedge B\text{—}Q\text{—}C \Rightarrow \exists X, P\text{—}X\text{—}B \wedge Q\text{—}X\text{—}A$$



Pasch's axiom

Axiom (Pasch)

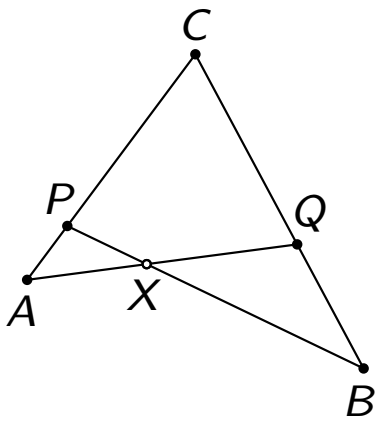
$$A\text{—}P\text{—}C \wedge B\text{—}Q\text{—}C \Rightarrow \exists X, P\text{—}X\text{—}B \wedge Q\text{—}X\text{—}A$$



Pasch's axiom

Axiom (Pasch)

$$A\text{—}P\text{—}C \wedge B\text{—}Q\text{—}C \Rightarrow \exists X, P\text{—}X\text{—}B \wedge Q\text{—}X\text{—}A$$



2-Dimensional Axiom

2-Dimensional Axiom

Axiom (Lower 2-Dimensional)

$$\exists ABC, \neg A\text{—}B\text{—}C \wedge \neg B\text{—}C\text{—}A \wedge \neg C\text{—}A\text{—}B$$

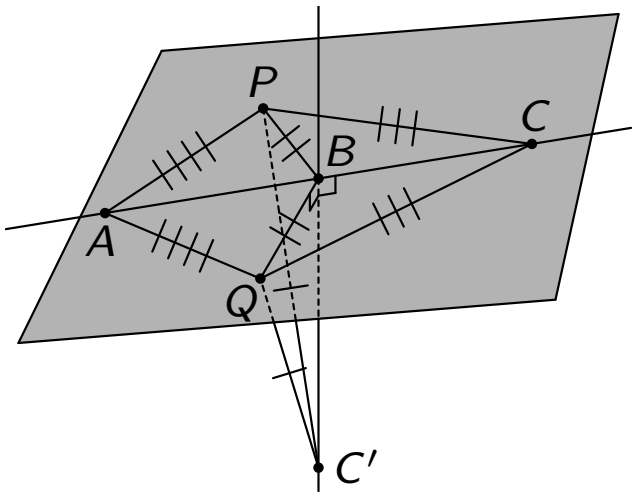
2-Dimensional Axiom

Axiom (Lower 2-Dimensional)

$$\exists ABC, \neg A\text{—}B\text{—}C \wedge \neg B\text{—}C\text{—}A \wedge \neg C\text{—}A\text{—}B$$

Axiom (Upper 2-Dimensional)

$$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$$
$$A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B$$

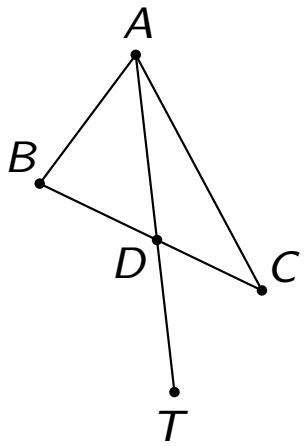


Euclid's axiom

Euclid's axiom

Axiom (Euclid)

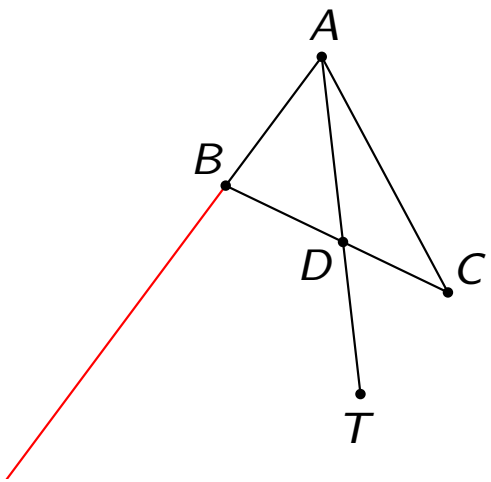
$$A\text{—}D\text{—}T \wedge B\text{—}D\text{—}C \wedge A \neq D \Rightarrow$$
$$\exists XY, A\text{—}B\text{—}X \wedge A\text{—}C\text{—}Y \wedge X\text{—}T\text{—}Y$$



Euclid's axiom

Axiom (Euclid)

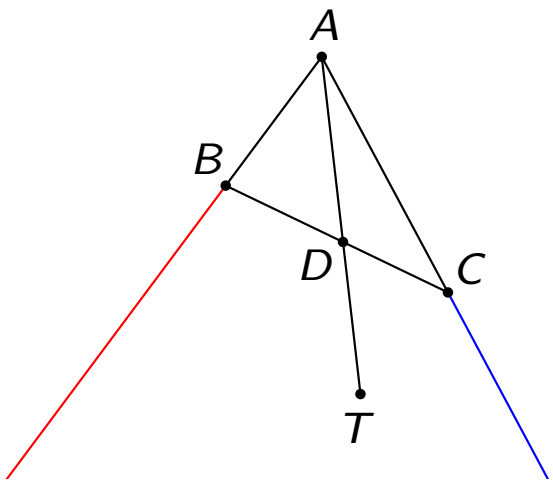
$$A\text{—}D\text{—}T \wedge B\text{—}D\text{—}C \wedge A \neq D \Rightarrow$$
$$\exists XY, A\text{—}B\text{—}X \wedge A\text{—}C\text{—}Y \wedge X\text{—}T\text{—}Y$$



Euclid's axiom

Axiom (Euclid)

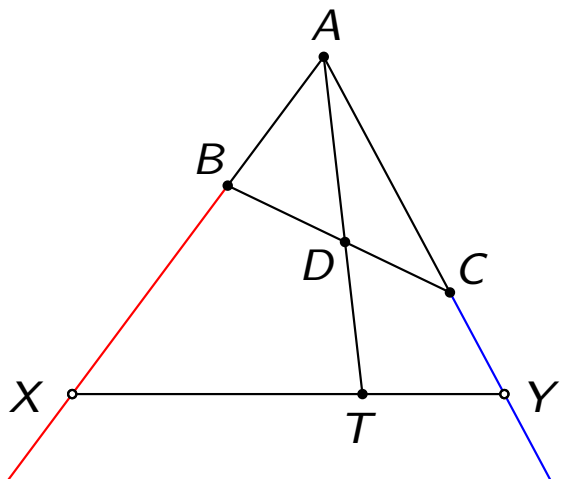
$$A\text{—}D\text{—}T \wedge B\text{—}D\text{—}C \wedge A \neq D \Rightarrow$$
$$\exists XY, A\text{—}B\text{—}X \wedge A\text{—}C\text{—}Y \wedge X\text{—}T\text{—}Y$$



Euclid's axiom

Axiom (Euclid)

$$A\text{—}D\text{—}T \wedge B\text{—}D\text{—}C \wedge A \neq D \Rightarrow$$
$$\exists XY, A\text{—}B\text{—}X \wedge A\text{—}C\text{—}Y \wedge X\text{—}T\text{—}Y$$



The axioms

Identity for betweenness	$A\text{—}B\text{—}A \Rightarrow A = B$
Transitivity for congruence	$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A\text{—}B\text{—}E \wedge BE \equiv CD$
Pasch	$A\text{—}P\text{—}C \wedge B\text{—}Q\text{—}C \Rightarrow \exists X, P\text{—}X\text{—}B \wedge Q\text{—}X\text{—}A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A\text{—}B\text{—}C \wedge A'\text{—}B'\text{—}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A\text{—}B\text{—}C \wedge \neg B\text{—}C\text{—}A \wedge \neg C\text{—}A\text{—}B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$ $A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B$
Euclid	$A\text{—}D\text{—}T \wedge B\text{—}D\text{—}C \wedge A \neq D \Rightarrow$ $\exists XY, A\text{—}B\text{—}X \wedge A\text{—}C\text{—}Y \wedge X\text{—}T\text{—}Y$
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow A\text{—}X\text{—}Y)) \Rightarrow$ $\exists B, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow X\text{—}B\text{—}Y)$

The axioms

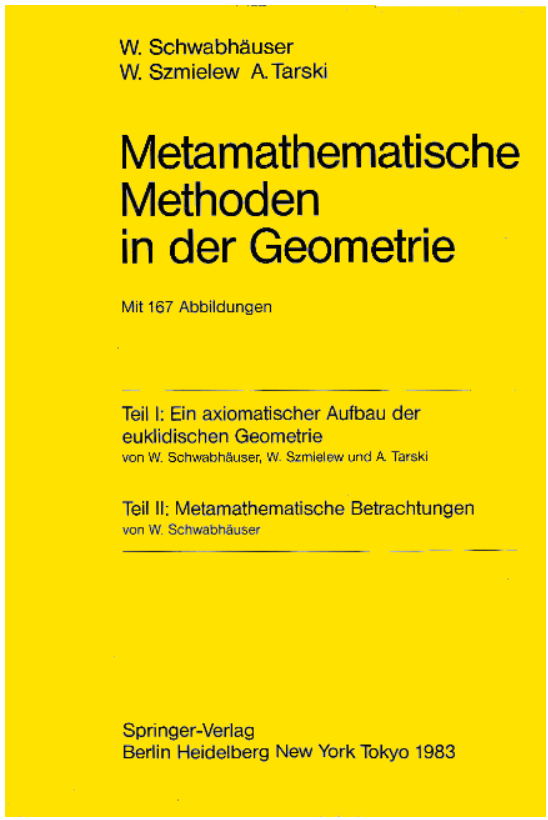
Identity for betweenness	$A\text{—}B\text{—}A \Rightarrow A = B$
Transitivity for congruence	$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A\text{—}B\text{—}E \wedge BE \equiv CD$
Pasch	$A\text{—}P\text{—}C \wedge B\text{—}Q\text{—}C \Rightarrow \exists X, P\text{—}X\text{—}B \wedge Q\text{—}X\text{—}A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A\text{—}B\text{—}C \wedge A'\text{—}B'\text{—}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A\text{—}B\text{—}C \wedge \neg B\text{—}C\text{—}A \wedge \neg C\text{—}A\text{—}B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$ $A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B$
Euclid	$A\text{—}D\text{—}T \wedge B\text{—}D\text{—}C \wedge A \neq D \Rightarrow$ $\exists XY, A\text{—}B\text{—}X \wedge A\text{—}C\text{—}Y \wedge X\text{—}T\text{—}Y$
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow A\text{—}X\text{—}Y)) \Rightarrow$ $\exists B, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow X\text{—}B\text{—}Y)$

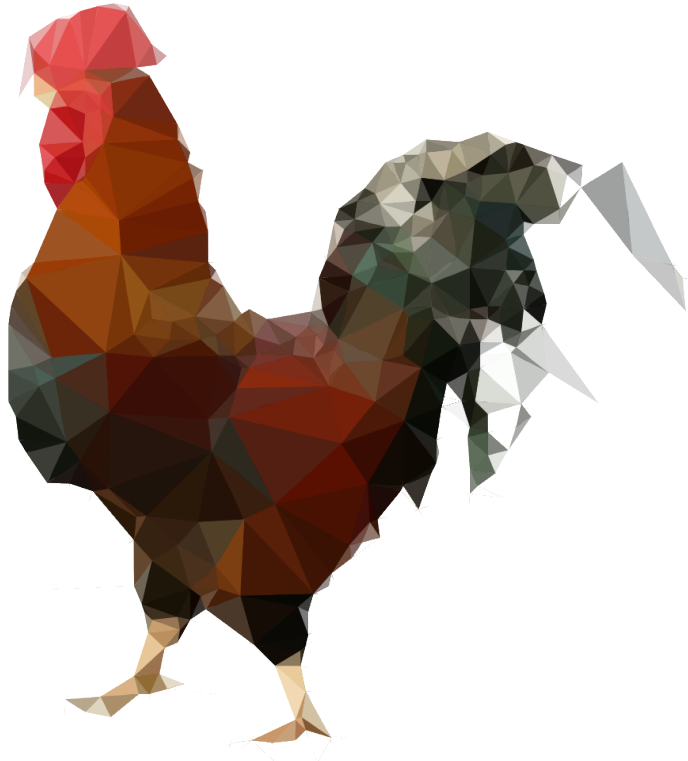
The axioms

Identity for betweenness	$A\text{—}B\text{—}A \Rightarrow A = B$
Transitivity for congruence	$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A\text{—}B\text{—}E \wedge BE \equiv CD$
Pasch	$A\text{—}P\text{—}C \wedge B\text{—}Q\text{—}C \Rightarrow \exists X, P\text{—}X\text{—}B \wedge Q\text{—}X\text{—}A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A\text{—}B\text{—}C \wedge A'\text{—}B'\text{—}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A\text{—}B\text{—}C \wedge \neg B\text{—}C\text{—}A \wedge \neg C\text{—}A\text{—}B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$ $A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B$
Euclid	$A\text{—}D\text{—}T \wedge B\text{—}D\text{—}C \wedge A \neq D \Rightarrow$ $\exists XY, A\text{—}B\text{—}X \wedge A\text{—}C\text{—}Y \wedge X\text{—}T\text{—}Y$
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow A\text{—}X\text{—}Y)) \Rightarrow$ $\exists B, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow X\text{—}B\text{—}Y)$
Point equality decidability	$X = Y \vee X \neq Y$

Overview of the formalization

Overview of the formalization



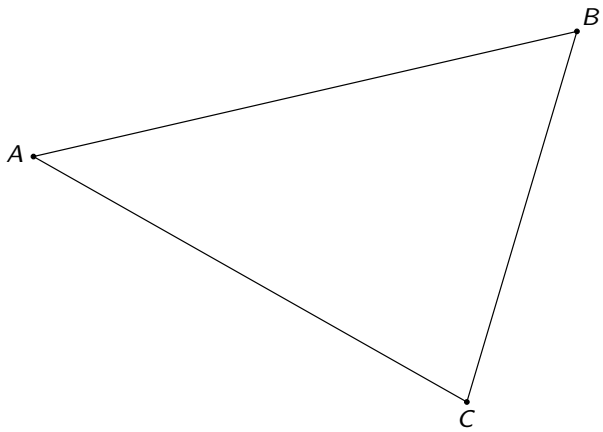


geocoq.github.io/GeoCoq/

An example of proof by computation

An example of proof by computation

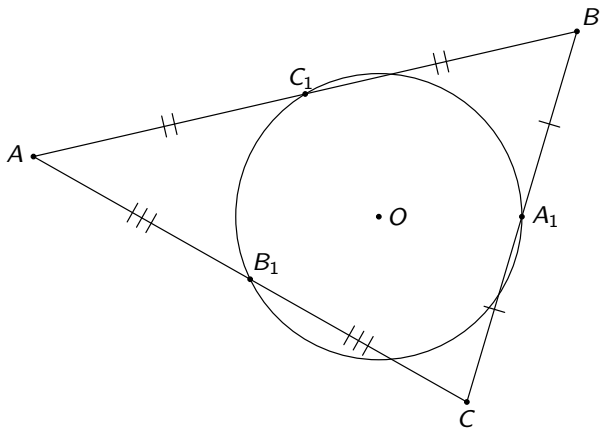
Our example is the nine-point circle theorem which states that the following nine points are concyclic:



An example of proof by computation

Our example is the nine-point circle theorem which states that the following nine points are concyclic:

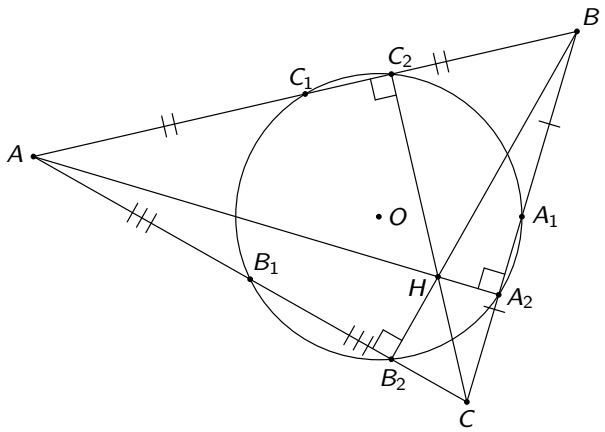
- The midpoints of each side of the triangle;



An example of proof by computation

Our example is the nine-point circle theorem which states that the following nine points are concyclic:

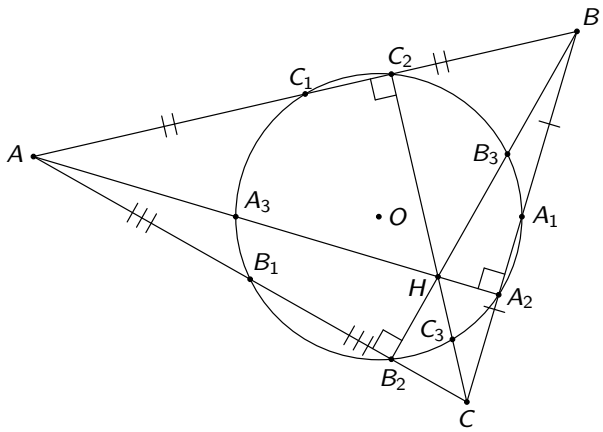
- The midpoints of each side of the triangle;
- The feet of each altitude;



An example of proof by computation

Our example is the nine-point circle theorem which states that the following nine points are concyclic:

- The midpoints of each side of the triangle;
- The feet of each altitude;
- The midpoints of the line-segments from each vertex of the triangle to the orthocenter.



A model of the theory

A model of the theory

- Points: $\mathbb{F}^2$ where $\mathbb{F}$ is a real closed field.

A model of the theory

- Points: $\mathbb{F}^2$ where $\mathbb{F}$ is a real closed field.
- $AB \equiv CD := (x_A - x_B)^2 + (y_A - y_B)^2 = (x_C - x_D)^2 + (y_C - y_D)^2$.

A model of the theory

- Points: $\mathbb{F}^2$ where $\mathbb{F}$ is a real closed field.
- $AB \equiv CD := (x_A - x_B)^2 + (y_A - y_B)^2 = (x_C - x_D)^2 + (y_C - y_D)^2$.
- $A\text{—}B\text{—}C := \exists k, 0 \leq k \leq 1 \wedge B - A = k(C - A)$.

Outline

1 Introduction

2 Tarski's system of geometry

3 Gupta's and Szczerba's contributions

4 An independent version of Tarski's system of geometry?

5 Conclusion

Gupta's axiom system

Gupta's axiom system

Identity for betweenness	$A\text{—}B\text{—}A \Rightarrow A = B$
Transitivity for congruence	$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A\text{—}B\text{—}E \wedge BE \equiv CD$
Pasch	$A\text{—}P\text{—}C \wedge B\text{—}Q\text{—}C \Rightarrow \exists X, P\text{—}X\text{—}B \wedge Q\text{—}X\text{—}A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A\text{—}B\text{—}C \wedge A'\text{—}B'\text{—}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A\text{—}B\text{—}C \wedge \neg B\text{—}C\text{—}A \wedge \neg C\text{—}A\text{—}B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$ $A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B$
Euclid	$A\text{—}D\text{—}T \wedge B\text{—}D\text{—}C \wedge A \neq D \Rightarrow$ $\exists XY, A\text{—}B\text{—}X \wedge A\text{—}C\text{—}Y \wedge X\text{—}T\text{—}Y$
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow A\text{—}X\text{—}Y)) \Rightarrow$ $\exists B, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow X\text{—}B\text{—}Y)$

Gupta's axiom system

Transitivity for betweenness	$A\text{—}B\text{—}D \wedge B\text{—}C\text{—}D \Rightarrow A\text{—}B\text{—}C$
Transitivity for congruence	$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A\text{—}B\text{—}E \wedge BE \equiv CD$
Pasch	$A\text{—}P\text{—}C \wedge B\text{—}Q\text{—}C \Rightarrow \exists X, P\text{—}X\text{—}B \wedge Q\text{—}X\text{—}A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A\text{—}B\text{—}C \wedge A'\text{—}B'\text{—}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A\text{—}B\text{—}C \wedge \neg B\text{—}C\text{—}A \wedge \neg C\text{—}A\text{—}B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$ $A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B$
Euclid	$A\text{—}D\text{—}T \wedge B\text{—}D\text{—}C \wedge A \neq D \Rightarrow$ $\exists XY, A\text{—}B\text{—}X \wedge A\text{—}C\text{—}Y \wedge X\text{—}T\text{—}Y$
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow A\text{—}X\text{—}Y)) \Rightarrow$ $\exists B, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow X\text{—}B\text{—}Y)$

Gupta's axiom system

Transitivity for betweenness	$A\text{—}B\text{—}D \wedge B\text{—}C\text{—}D \Rightarrow A\text{—}B\text{—}C$
Transitivity for congruence	$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A\text{—}B\text{—}E \wedge BE \equiv CD$
Pasch	$A\text{—}P\text{—}C \wedge B\text{—}Q\text{—}C \Rightarrow \exists X, P\text{—}X\text{—}B \wedge Q\text{—}X\text{—}A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A\text{—}B\text{—}C \wedge A'\text{—}B'\text{—}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A\text{—}B\text{—}C \wedge \neg B\text{—}C\text{—}A \wedge \neg C\text{—}A\text{—}B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$ $A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B$
Euclid	$A\text{—}D\text{—}T \wedge B\text{—}D\text{—}C \wedge A \neq D \Rightarrow$ $\exists XY, A\text{—}B\text{—}X \wedge A\text{—}C\text{—}Y \wedge X\text{—}T\text{—}Y$
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow A\text{—}X\text{—}Y)) \Rightarrow$ $\exists B, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow X\text{—}B\text{—}Y)$

Gupta's axiom system

Transitivity for betweenness	$A\text{—}B\text{—}D \wedge B\text{—}C\text{—}D \Rightarrow A\text{—}B\text{—}C$
Transitivity for congruence	$AB \equiv EF \wedge CD \equiv EF \Rightarrow AB \equiv CD$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A\text{—}B\text{—}E \wedge BE \equiv CD$
Pasch	$A\text{—}P\text{—}C \wedge B\text{—}Q\text{—}C \Rightarrow \exists X, P\text{—}X\text{—}B \wedge Q\text{—}X\text{—}A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A\text{—}B\text{—}C \wedge A'\text{—}B'\text{—}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A\text{—}B\text{—}C \wedge \neg B\text{—}C\text{—}A \wedge \neg C\text{—}A\text{—}B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$ $A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B$
Euclid	$A\text{—}D\text{—}T \wedge B\text{—}D\text{—}C \wedge A \neq D \Rightarrow$ $\exists XY, A\text{—}B\text{—}X \wedge A\text{—}C\text{—}Y \wedge X\text{—}T\text{—}Y$
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow A\text{—}X\text{—}Y)) \Rightarrow$ $\exists B, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow X\text{—}B\text{—}Y)$

Gupta's axiom system

Transitivity for betweenness	$A\text{—}B\text{—}D \wedge B\text{—}C\text{—}D \Rightarrow A\text{—}B\text{—}C$
Transitivity for congruence	$AB \equiv EF \wedge CD \equiv EF \Rightarrow AB \equiv CD$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A\text{—}B\text{—}E \wedge BE \equiv CD$
Pasch	$A\text{—}P\text{—}C \wedge B\text{—}Q\text{—}C \Rightarrow \exists X, P\text{—}X\text{—}B \wedge Q\text{—}X\text{—}A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A\text{—}B\text{—}C \wedge A'\text{—}B'\text{—}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A\text{—}B\text{—}C \wedge \neg B\text{—}C\text{—}A \wedge \neg C\text{—}A\text{—}B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$ $A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B$
Euclid	$A\text{—}D\text{—}T \wedge B\text{—}D\text{—}C \wedge A \neq D \Rightarrow$ $\exists XY, A\text{—}B\text{—}X \wedge A\text{—}C\text{—}Y \wedge X\text{—}T\text{—}Y$
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow A\text{—}X\text{—}Y)) \Rightarrow$ $\exists B, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow X\text{—}B\text{—}Y)$

Gupta's axiom system

Transitivity for betweenness	$A\text{—}B\text{—}D \wedge B\text{—}C\text{—}D \Rightarrow A\text{—}B\text{—}C$
Transitivity for congruence	$AB \equiv EF \wedge CD \equiv EF \Rightarrow AB \equiv CD$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A\text{—}B\text{—}E \wedge BE \equiv CD$
Pasch	$A\text{—}P\text{—}C \wedge B\text{—}Q\text{—}C \Rightarrow \exists X, P\text{—}X\text{—}B \wedge Q\text{—}X\text{—}A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A\text{—}B\text{—}C \wedge A'\text{—}B'\text{—}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A\text{—}B\text{—}C \wedge \neg B\text{—}C\text{—}A \wedge \neg C\text{—}A\text{—}B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \wedge$ $A \neq B \wedge B \neq C \wedge A \neq C \Rightarrow$ $A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B$
Euclid	$A\text{—}D\text{—}T \wedge B\text{—}D\text{—}C \wedge A \neq D \Rightarrow$ $\exists XY, A\text{—}B\text{—}X \wedge A\text{—}C\text{—}Y \wedge X\text{—}T\text{—}Y$
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow A\text{—}X\text{—}Y)) \Rightarrow$ $\exists B, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow X\text{—}B\text{—}Y)$

Gupta's axiom system

Transitivity for betweenness	$A\text{—}B\text{—}D \wedge B\text{—}C\text{—}D \Rightarrow A\text{—}B\text{—}C$
Transitivity for congruence	$AB \equiv EF \wedge CD \equiv EF \Rightarrow AB \equiv CD$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A\text{—}B\text{—}E \wedge BE \equiv CD$
Pasch	$A\text{—}P\text{—}C \wedge B\text{—}Q\text{—}C \Rightarrow \exists X, P\text{—}X\text{—}B \wedge Q\text{—}X\text{—}A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A\text{—}B\text{—}C \wedge A'\text{—}B'\text{—}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A\text{—}B\text{—}C \wedge \neg B\text{—}C\text{—}A \wedge \neg C\text{—}A\text{—}B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \wedge$ $A \neq B \wedge B \neq C \wedge A \neq C \Rightarrow$ $A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B$
Euclid	$A\text{—}D\text{—}T \wedge B\text{—}D\text{—}C \wedge A \neq D \Rightarrow$ $\exists XY, A\text{—}B\text{—}X \wedge A\text{—}C\text{—}Y \wedge X\text{—}T\text{—}Y$
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow A\text{—}X\text{—}Y)) \Rightarrow$ $\exists B, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow X\text{—}B\text{—}Y)$

Gupta's axiom system

Transitivity for betweenness	$A\text{—}B\text{—}D \wedge B\text{—}C\text{—}D \Rightarrow A\text{—}B\text{—}C$
Transitivity for congruence	$AB \equiv EF \wedge CD \equiv EF \Rightarrow AB \equiv CD$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A\text{—}B\text{—}E \wedge BE \equiv CD$
Pasch	$A\text{—}P\text{—}C \wedge B\text{—}Q\text{—}C \Rightarrow \exists X, P\text{—}X\text{—}B \wedge Q\text{—}X\text{—}A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A\text{—}B\text{—}C \wedge A'\text{—}B'\text{—}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A\text{—}B\text{—}C \wedge \neg B\text{—}C\text{—}A \wedge \neg C\text{—}A\text{—}B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \wedge$ $A \neq B \wedge B \neq C \wedge A \neq C \Rightarrow$ $A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B$
Euclid	$A\text{—}D\text{—}T \wedge B\text{—}D\text{—}C \wedge A \neq D \Rightarrow$ $\exists XY, A\text{—}B\text{—}X \wedge A\text{—}C\text{—}Y \wedge X\text{—}T\text{—}Y$
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow A\text{—}X\text{—}Y)) \Rightarrow$ $\exists B, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow$ $X\text{—}B\text{—}Y \vee X = B \vee B = Y)$

Gupta's axiom system

Transitivity for betweenness	$A\text{—}B\text{—}D \wedge B\text{—}C\text{—}D \Rightarrow A\text{—}B\text{—}C$
Transitivity for congruence	$AB \equiv EF \wedge CD \equiv EF \Rightarrow AB \equiv CD$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A\text{—}B\text{—}E \wedge BE \equiv CD$
Pasch	$A\text{—}P\text{—}C \wedge B\text{—}Q\text{—}C \Rightarrow \exists X, P\text{—}X\text{—}B \wedge Q\text{—}X\text{—}A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A\text{—}B\text{—}C \wedge A'\text{—}B'\text{—}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A\text{—}B\text{—}C \wedge \neg B\text{—}C\text{—}A \wedge \neg C\text{—}A\text{—}B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \wedge$ $A \neq B \wedge B \neq C \wedge A \neq C \Rightarrow$ $A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B$
Euclid	$A\text{—}D\text{—}T \wedge B\text{—}D\text{—}C \wedge A \neq D \Rightarrow$ $\exists XY, A\text{—}B\text{—}X \wedge A\text{—}C\text{—}Y \wedge X\text{—}T\text{—}Y$
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow A\text{—}X\text{—}Y)) \Rightarrow$ $\exists B, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow$ $X\text{—}B\text{—}Y \vee X = B \vee B = Y)$

Gupta's axiom system

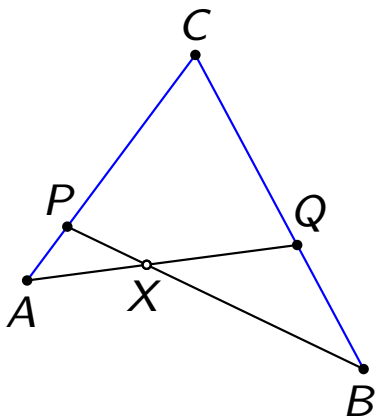
Transitivity for betweenness	$A\text{—}B\text{—}D \wedge B\text{—}C\text{—}D \Rightarrow A\text{—}B\text{—}C$
Transitivity for congruence	$AB \equiv EF \wedge CD \equiv EF \Rightarrow AB \equiv CD$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A\text{—}B\text{—}E \wedge BE \equiv CD$
Pasch	$A\text{—}X\text{—}Q \wedge C\text{—}Q\text{—}B \Rightarrow \exists P, A\text{—}P\text{—}C \wedge B\text{—}X\text{—}P$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A\text{—}B\text{—}C \wedge A'\text{—}B'\text{—}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A\text{—}B\text{—}C \wedge \neg B\text{—}C\text{—}A \wedge \neg C\text{—}A\text{—}B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \wedge$ $A \neq B \wedge B \neq C \wedge A \neq C \Rightarrow$ $A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B$
Euclid	$A\text{—}D\text{—}T \wedge B\text{—}D\text{—}C \wedge A \neq D \Rightarrow$ $\exists XY, A\text{—}B\text{—}X \wedge A\text{—}C\text{—}Y \wedge X\text{—}T\text{—}Y$
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow A\text{—}X\text{—}Y)) \Rightarrow$ $\exists B, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow$ $X\text{—}B\text{—}Y \vee X = B \vee B = Y)$

Versions of Pasch's axiom

Versions of Pasch's axiom

Axiom (Inner Pasch)

$$A\text{—}P\text{—}C \wedge B\text{—}Q\text{—}C \Rightarrow \exists X, P\text{—}X\text{—}B \wedge Q\text{—}X\text{—}A$$



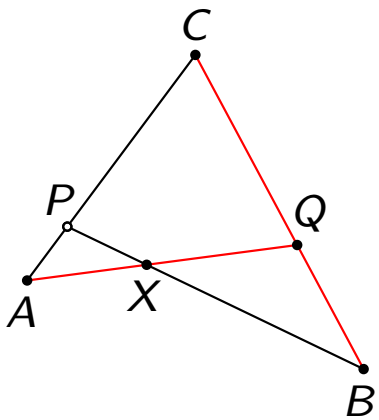
Versions of Pasch's axiom

Axiom (Inner Pasch)

$$A\text{—}P\text{—}C \wedge B\text{—}Q\text{—}C \Rightarrow \exists X, P\text{—}X\text{—}B \wedge Q\text{—}X\text{—}A$$

Axiom (Outer Pasch)

$$A\text{—}X\text{—}Q \wedge C\text{—}Q\text{—}B \Rightarrow \exists P, A\text{—}P\text{—}C \wedge B\text{—}X\text{—}P$$



Gupta's independence model for Pasch

Gupta's independence model for Pasch

- Points: $\mathbb{F}^2$ where $\mathbb{F}$ is a real closed field.

Gupta's independence model for Pasch

- Points: $\mathbb{F}^2$ where $\mathbb{F}$ is a real closed field.
- $AB \equiv CD := (x_A - x_B)^2 + (y_A - y_B)^2 = (x_C - x_D)^2 + (y_C - y_D)^2$.

Gupta's independence model for Pasch

- Points: $\mathbb{F}^2$ where $\mathbb{F}$ is a real closed field.
- $AB \equiv CD := (x_A - x_B)^2 + (y_A - y_B)^2 = (x_C - x_D)^2 + (y_C - y_D)^2$.
- $A\text{—}B\text{—}C := \exists k, 0 \leq k \leq 1 \wedge B - A = k(C - A)$ at the exception of the cases where $A = B$ and both A and C belong to the x-axis.

Szczerba's axiom system

Szczerba's axiom system

Identity for betweenness	
Inner transitivity for betweenness	
Outer transitivity for betweenness	
Transitivity for congruence	
Reflexivity for congruence	
Identity for congruence	
Segment Construction	
Pasch	$A\text{—}X\text{—}Q \wedge C\text{—}Q\text{—}B \Rightarrow \exists P, A\text{—}P\text{—}C \wedge B\text{—}X\text{—}P$
Five-Segment	
Lower 2-Dimensional	
Upper 2-Dimensional	
Euclid	$\neg(A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B) \Rightarrow$ $\exists C_C, AC_C \equiv BC_C \wedge AC_C \equiv CC_C$
Continuity	

Szczerba's axiom system

Identity for betweenness	
Inner transitivity for betweenness	
Outer transitivity for betweenness	
Transitivity for congruence	
Reflexivity for congruence	
Identity for congruence	
Segment Construction	
Pasch	$A—X—Q \wedge C—Q—B \Rightarrow \exists P, A—P—C \wedge B—X—P$
Five-Segment	
Lower 2-Dimensional	
Upper 2-Dimensional	
Euclid	$\neg(A—B—C \vee B—C—A \vee C—A—B) \Rightarrow$ $\exists C_C, AC_C \equiv BC_C \wedge AC_C \equiv CC_C$
Continuity	

Triangle circumscription principle

Triangle circumscription principle

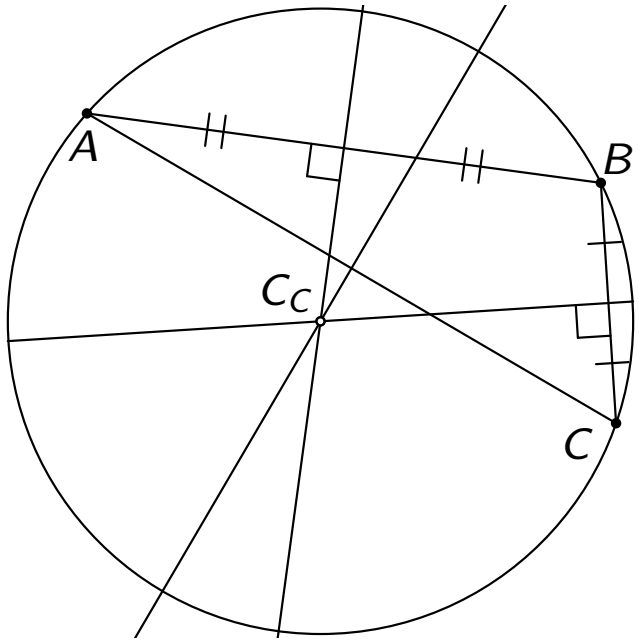
Axiom (Triangle circumscription principle)

$$\neg(A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B) \Rightarrow$$
$$\exists C_C, AC_C \equiv BC_C \wedge AC_C \equiv CC_C$$

Triangle circumscription principle

Axiom (Triangle circumscription principle)

$$\neg(A{-}B{-}C \vee B{-}C{-}A \vee C{-}A{-}B) \Rightarrow$$
$$\exists C_C, AC_C \equiv BC_C \wedge AC_C \equiv CC_C$$



Outline

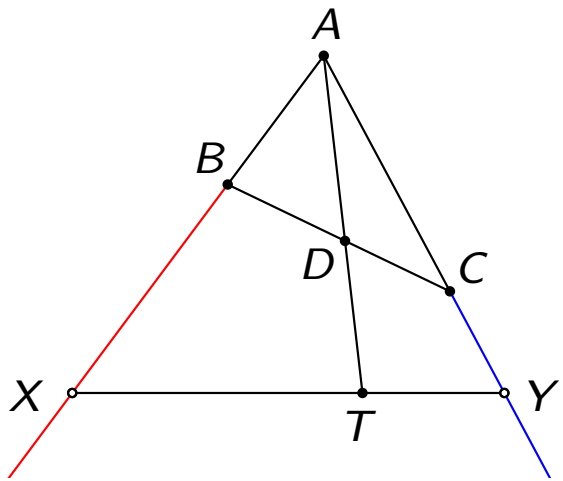
1. Introduction
2. Tarski's system of geometry
3. Gupta's and Szczerba's contributions
4. An independent version of Tarski's system of geometry?
5. Conclusion

Euclid's axiom

Euclid's axiom

Axiom (Euclid)

$$A\text{—}D\text{—}T \wedge B\text{—}D\text{—}C \wedge A \neq D \Rightarrow$$
$$\exists XY, A\text{—}B\text{—}X \wedge A\text{—}C\text{—}Y \wedge X\text{—}T\text{—}Y$$



Syntactic proof

Identity for betweenness	$A\text{—}B\text{—}A \Rightarrow A = B$
Transitivity for congruence	$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A\text{—}B\text{—}E \wedge BE \equiv CD$
Pasch	$A\text{—}P\text{—}C \wedge B\text{—}Q\text{—}C \Rightarrow \exists X, P\text{—}X\text{—}B \wedge Q\text{—}X\text{—}A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A\text{—}B\text{—}C \wedge A'\text{—}B'\text{—}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A\text{—}B\text{—}C \wedge \neg B\text{—}C\text{—}A \wedge \neg C\text{—}A\text{—}B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$ $A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B$
Euclid	$A\text{—}D\text{—}T \wedge B\text{—}D\text{—}C \wedge A \neq D \Rightarrow$ $\exists XY, A\text{—}B\text{—}X \wedge A\text{—}C\text{—}Y \wedge X\text{—}T\text{—}Y$
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, X \in \Xi \wedge Y \in \Upsilon \Rightarrow A\text{—}X\text{—}Y)) \Rightarrow$ $\exists B, (\forall XY, X \in \Xi \wedge Y \in \Upsilon \Rightarrow X\text{—}B\text{—}Y)$

Syntactic proof

Identity for betweenness	$A\text{—}B\text{—}A \Rightarrow A = B$
Transitivity for congruence	$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A\text{—}B\text{—}E \wedge BE \equiv CD$
Pasch	$A\text{—}P\text{—}C \wedge B\text{—}Q\text{—}C \Rightarrow \exists X, P\text{—}X\text{—}B \wedge Q\text{—}X\text{—}A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A\text{—}B\text{—}C \wedge A'\text{—}B'\text{—}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A\text{—}B\text{—}C \wedge \neg B\text{—}C\text{—}A \wedge \neg C\text{—}A\text{—}B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$ $A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B$
Euclid	$A\text{—}D\text{—}T \wedge B\text{—}D\text{—}C \wedge A \neq D \Rightarrow$ $\exists XY, A\text{—}B\text{—}X \wedge A\text{—}C\text{—}Y \wedge X\text{—}T\text{—}Y$

Syntactic proof

Identity for betweenness	$A\text{—}B\text{—}A \Rightarrow A = B$
Transitivity for congruence	$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A\text{—}B\text{—}E \wedge BE \equiv CD$
Pasch	$A\text{—}P\text{—}C \wedge B\text{—}Q\text{—}C \Rightarrow \exists X, P\text{—}X\text{—}B \wedge Q\text{—}X\text{—}A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A\text{—}B\text{—}C \wedge A'\text{—}B'\text{—}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A\text{—}B\text{—}C \wedge \neg B\text{—}C\text{—}A \wedge \neg C\text{—}A\text{—}B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$ $A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B$
Euclid	$A\text{—}D\text{—}T \wedge B\text{—}D\text{—}C \wedge A \neq D \Rightarrow$ $\exists XY, A\text{—}B\text{—}X \wedge A\text{—}C\text{—}Y \wedge X\text{—}T\text{—}Y$

Syntactic proof

Identity for betweenness	$A\text{—}B\text{—}A \Rightarrow A = B$
Transitivity for congruence	$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A\text{—}B\text{—}E \wedge BE \equiv CD$
Pasch	$A\text{—}P\text{—}C \wedge B\text{—}Q\text{—}C \Rightarrow \exists X, P\text{—}X\text{—}B \wedge Q\text{—}X\text{—}A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A\text{—}B\text{—}C \wedge A'\text{—}B'\text{—}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A\text{—}B\text{—}C \wedge \neg B\text{—}C\text{—}A \wedge \neg C\text{—}A\text{—}B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$ $A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B$
Euclid	$A\text{—}D\text{—}T \wedge B\text{—}D\text{—}C \wedge A \neq D \Rightarrow$ $\exists XY, A\text{—}B\text{—}X \wedge A\text{—}C\text{—}Y \wedge X\text{—}T\text{—}Y$

Syntactic proof

Identity for betweenness	$A\text{—}B\text{—}A \Rightarrow A = B$
Transitivity for congruence	$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A\text{—}B\text{—}E \wedge BE \equiv CD$
Pasch	$A\text{—}P\text{—}C \wedge B\text{—}Q\text{—}C \Rightarrow \exists X, P\text{—}X\text{—}B \wedge Q\text{—}X\text{—}A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A\text{—}B\text{—}C \wedge A'\text{—}B'\text{—}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A\text{—}B\text{—}C \wedge \neg B\text{—}C\text{—}A \wedge \neg C\text{—}A\text{—}B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$ $A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B$
Euclid	$A\text{—}D\text{—}T \wedge B\text{—}D\text{—}C \wedge A \neq D \Rightarrow$ $\exists XY, A\text{—}B\text{—}X \wedge A\text{—}C\text{—}Y \wedge X\text{—}T\text{—}Y$

Syntactic proof

$A\text{–}B\text{–}A \Rightarrow A = B$

$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$

$AB \equiv BA$

$AB \equiv CC \Rightarrow A = B$

$\exists E, A\text{–}B\text{–}E \wedge BE \equiv CD$

$A\text{–}P\text{–}C \wedge B\text{–}Q\text{–}C \Rightarrow \exists X, P\text{–}X\text{–}B \wedge Q\text{–}X\text{–}A$

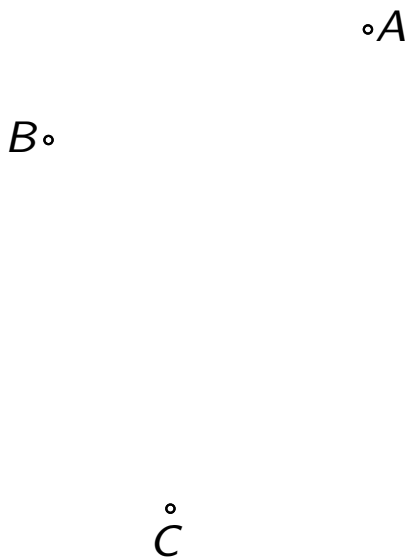
$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$
$AD \equiv A'D' \wedge BD \equiv B'D' \wedge$
$A\text{–}B\text{–}C \wedge A'\text{–}B'\text{–}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$

$\exists ABC, \neg A\text{–}B\text{–}C \wedge \neg B\text{–}C\text{–}A \wedge \neg C\text{–}A\text{–}B$

$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$
$A\text{–}B\text{–}C \vee B\text{–}C\text{–}A \vee C\text{–}A\text{–}B$

$A\text{–}D\text{–}T \wedge B\text{–}D\text{–}C \wedge A \neq D \Rightarrow$
$\exists XY, A\text{–}B\text{–}X \wedge A\text{–}C\text{–}Y \wedge X\text{–}T\text{–}Y$

Syntactic proof



$A{-}B{-}A \Rightarrow A = B$

$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$

$AB \equiv BA$

$AB \equiv CC \Rightarrow A = B$

$\exists E, A{-}B{-}E \wedge BE \equiv CD$

$A{-}P{-}C \wedge B{-}Q{-}C \Rightarrow \exists X, P{-}X{-}B \wedge Q{-}X{-}A$

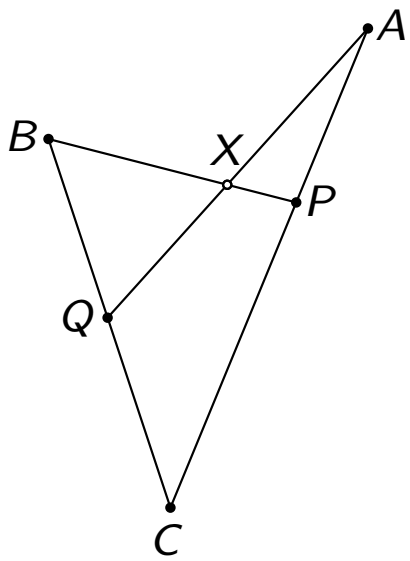
$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$
$AD \equiv A'D' \wedge BD \equiv B'D' \wedge$
$A{-}B{-}C \wedge A'{-}B'{-}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$

$\exists ABC, \neg A{-}B{-}C \wedge \neg B{-}C{-}A \wedge \neg C{-}A{-}B$

$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$
$A{-}B{-}C \vee B{-}C{-}A \vee C{-}A{-}B$

$A{-}D{-}T \wedge B{-}D{-}C \wedge A \neq D \Rightarrow$
$\exists XY, A{-}B{-}X \wedge A{-}C{-}Y \wedge X{-}T{-}Y$

Syntactic proof



$A\text{–}B\text{–}A \Rightarrow A = B$

$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$

$AB \equiv BA$

$AB \equiv CC \Rightarrow A = B$

$\exists E, A\text{–}B\text{–}E \wedge BE \equiv CD$

$A\text{–}P\text{–}C \wedge B\text{–}Q\text{–}C \Rightarrow \exists X, P\text{–}X\text{–}B \wedge Q\text{–}X\text{–}A$

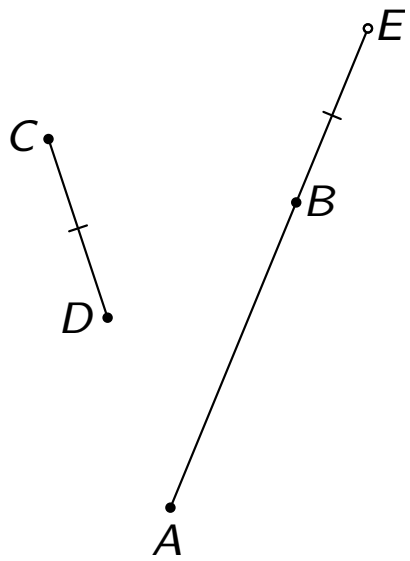
$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$
$AD \equiv A'D' \wedge BD \equiv B'D' \wedge$
$A\text{–}B\text{–}C \wedge A'\text{–}B'\text{–}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$

$\exists ABC, \neg A\text{–}B\text{–}C \wedge \neg B\text{–}C\text{–}A \wedge \neg C\text{–}A\text{–}B$

$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$
$A\text{–}B\text{–}C \vee B\text{–}C\text{–}A \vee C\text{–}A\text{–}B$

$A\text{–}D\text{–}T \wedge B\text{–}D\text{–}C \wedge A \neq D \Rightarrow$
$\exists XY, A\text{–}B\text{–}X \wedge A\text{–}C\text{–}Y \wedge X\text{–}T\text{–}Y$

Syntactic proof



A–B–$A \Rightarrow A = B$

$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$

$AB \equiv BA$

$AB \equiv CC \Rightarrow A = B$

$\exists E, A$–B–$E \wedge BE \equiv CD$

A–P–$C \wedge B$–Q–$C \Rightarrow \exists X, P$–$X$–$B \wedge Q$–$X$–$A$

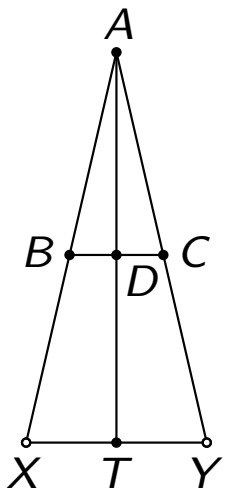
$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$
$AD \equiv A'D' \wedge BD \equiv B'D' \wedge$
A–B–$C \wedge A'$–B'–$C' \wedge A \neq B \Rightarrow CD \equiv C'D'$

$\exists ABC, \neg A$–B–$C \wedge \neg B$–C–$A \wedge \neg C$–A–B

$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$
A–B–$C \vee B$–C–$A \vee C$–A–B

A–D–$T \wedge B$–D–$C \wedge A \neq D \Rightarrow$
$\exists XY, A$–B–$X \wedge A$–C–$Y \wedge X$–T–Y

Syntactic proof



$A\text{–}B\text{–}A \Rightarrow A = B$

$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$

$AB \equiv BA$

$AB \equiv CC \Rightarrow A = B$

$\exists E, A\text{–}B\text{–}E \wedge BE \equiv CD$

$A\text{–}P\text{–}C \wedge B\text{–}Q\text{–}C \Rightarrow \exists X, P\text{–}X\text{–}B \wedge Q\text{–}X\text{–}A$

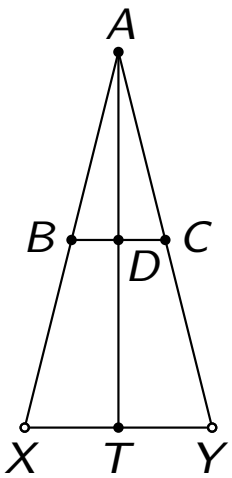
$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$
$AD \equiv A'D' \wedge BD \equiv B'D' \wedge$
$A\text{–}B\text{–}C \wedge A'\text{–}B'\text{–}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$

$\exists ABC, \neg A\text{–}B\text{–}C \wedge \neg B\text{–}C\text{–}A \wedge \neg C\text{–}A\text{–}B$

$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$
$A\text{–}B\text{–}C \vee B\text{–}C\text{–}A \vee C\text{–}A\text{–}B$

$A\text{–}D\text{–}T \wedge B\text{–}D\text{–}C \wedge A \neq D \Rightarrow$
$\exists XY, A\text{–}B\text{–}X \wedge A\text{–}C\text{–}Y \wedge X\text{–}T\text{–}Y$

Syntactic proof



$A\text{–}B\text{–}A \Rightarrow A = B$

$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$

$AB \equiv BA$

$AB \equiv CC \Rightarrow A = B$

$\exists E, A\text{–}B\text{–}E \wedge BE \equiv CD$

$A\text{–}P\text{–}C \wedge B\text{–}Q\text{–}C \Rightarrow \exists X, P\text{–}X\text{–}B \wedge Q\text{–}X\text{–}A$

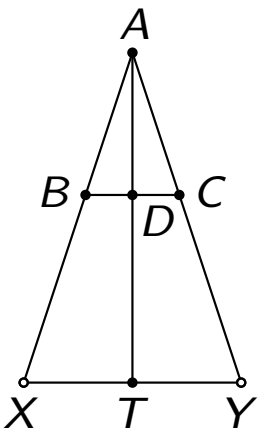
$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$
$AD \equiv A'D' \wedge BD \equiv B'D' \wedge$
$A\text{–}B\text{–}C \wedge A'\text{–}B'\text{–}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$

$\exists ABC, \neg A\text{–}B\text{–}C \wedge \neg B\text{–}C\text{–}A \wedge \neg C\text{–}A\text{–}B$

$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$
$A\text{–}B\text{–}C \vee B\text{–}C\text{–}A \vee C\text{–}A\text{–}B$

$A\text{–}D\text{–}T \wedge B\text{–}D\text{–}C \wedge A \neq D \Rightarrow$
$\exists XY, A\text{–}B\text{–}X \wedge A\text{–}C\text{–}Y \wedge X\text{–}T\text{–}Y$

Syntactic proof



$A\text{–}B\text{–}A \Rightarrow A = B$

$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$

$AB \equiv BA$

$AB \equiv CC \Rightarrow A = B$

$\exists E, A\text{–}B\text{–}E \wedge BE \equiv CD$

$A\text{–}P\text{–}C \wedge B\text{–}Q\text{–}C \Rightarrow \exists X, P\text{–}X\text{–}B \wedge Q\text{–}X\text{–}A$

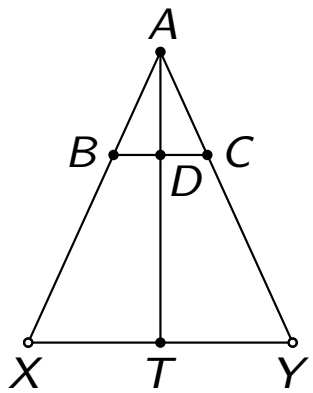
$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$
$AD \equiv A'D' \wedge BD \equiv B'D' \wedge$
$A\text{–}B\text{–}C \wedge A'\text{–}B'\text{–}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$

$\exists ABC, \neg A\text{–}B\text{–}C \wedge \neg B\text{–}C\text{–}A \wedge \neg C\text{–}A\text{–}B$

$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$
$A\text{–}B\text{–}C \vee B\text{–}C\text{–}A \vee C\text{–}A\text{–}B$

$A\text{–}D\text{–}T \wedge B\text{–}D\text{–}C \wedge A \neq D \Rightarrow$
$\exists XY, A\text{–}B\text{–}X \wedge A\text{–}C\text{–}Y \wedge X\text{–}T\text{–}Y$

Syntactic proof



$A\text{–}B\text{–}A \Rightarrow A = B$

$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$

$AB \equiv BA$

$AB \equiv CC \Rightarrow A = B$

$\exists E, A\text{–}B\text{–}E \wedge BE \equiv CD$

$A\text{–}P\text{–}C \wedge B\text{–}Q\text{–}C \Rightarrow \exists X, P\text{–}X\text{–}B \wedge Q\text{–}X\text{–}A$

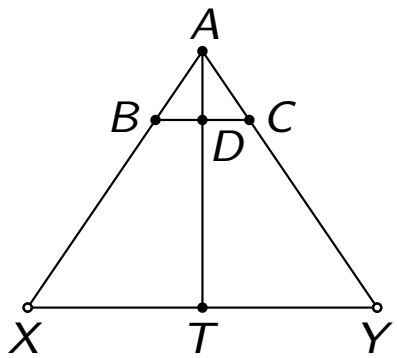
$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$
$AD \equiv A'D' \wedge BD \equiv B'D' \wedge$
$A\text{–}B\text{–}C \wedge A'\text{–}B'\text{–}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$

$\exists ABC, \neg A\text{–}B\text{–}C \wedge \neg B\text{–}C\text{–}A \wedge \neg C\text{–}A\text{–}B$

$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$
$A\text{–}B\text{–}C \vee B\text{–}C\text{–}A \vee C\text{–}A\text{–}B$

$A\text{–}D\text{–}T \wedge B\text{–}D\text{–}C \wedge A \neq D \Rightarrow$
$\exists XY, A\text{–}B\text{–}X \wedge A\text{–}C\text{–}Y \wedge X\text{–}T\text{–}Y$

Syntactic proof



$A\text{–}B\text{–}A \Rightarrow A = B$

$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$

$AB \equiv BA$

$AB \equiv CC \Rightarrow A = B$

$\exists E, A\text{–}B\text{–}E \wedge BE \equiv CD$

$A\text{–}P\text{–}C \wedge B\text{–}Q\text{–}C \Rightarrow \exists X, P\text{–}X\text{–}B \wedge Q\text{–}X\text{–}A$

$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$
$AD \equiv A'D' \wedge BD \equiv B'D' \wedge$
$A\text{–}B\text{–}C \wedge A'\text{–}B'\text{–}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$

$\exists ABC, \neg A\text{–}B\text{–}C \wedge \neg B\text{–}C\text{–}A \wedge \neg C\text{–}A\text{–}B$

$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$
$A\text{–}B\text{–}C \vee B\text{–}C\text{–}A \vee C\text{–}A\text{–}B$

$A\text{–}D\text{–}T \wedge B\text{–}D\text{–}C \wedge A \neq D \Rightarrow$
$\exists XY, A\text{–}B\text{–}X \wedge A\text{–}C\text{–}Y \wedge X\text{–}T\text{–}Y$

Syntactic proof

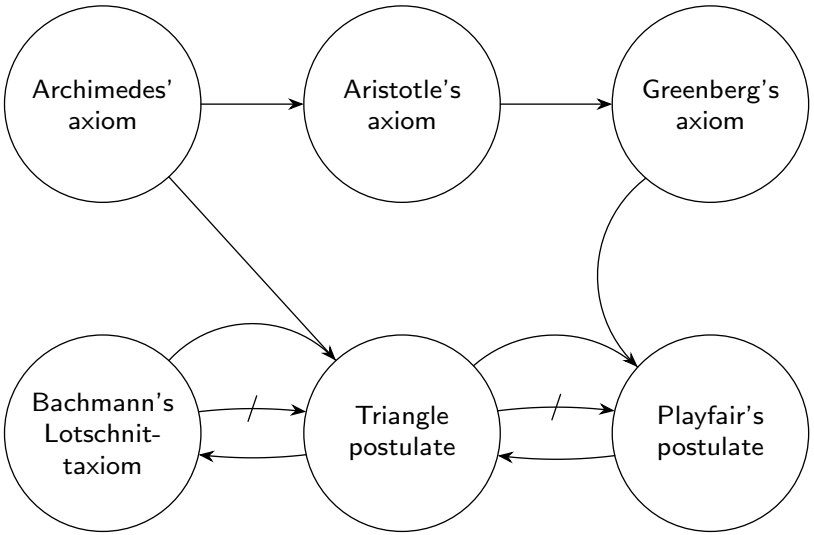
Identity for betweenness	$A\text{—}B\text{—}A \Rightarrow A = B$
Transitivity for congruence	$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A\text{—}B\text{—}E \wedge BE \equiv CD$
Pasch	$A\text{—}P\text{—}C \wedge B\text{—}Q\text{—}C \Rightarrow \exists X, P\text{—}X\text{—}B \wedge Q\text{—}X\text{—}A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A\text{—}B\text{—}C \wedge A'\text{—}B'\text{—}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A\text{—}B\text{—}C \wedge \neg B\text{—}C\text{—}A \wedge \neg C\text{—}A\text{—}B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$ $A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B$
Euclid	$A\text{—}D\text{—}T \wedge B\text{—}D\text{—}C \wedge A \neq D \Rightarrow$ $\exists XY, A\text{—}B\text{—}X \wedge A\text{—}C\text{—}Y \wedge X\text{—}T\text{—}Y$

Syntactic proof

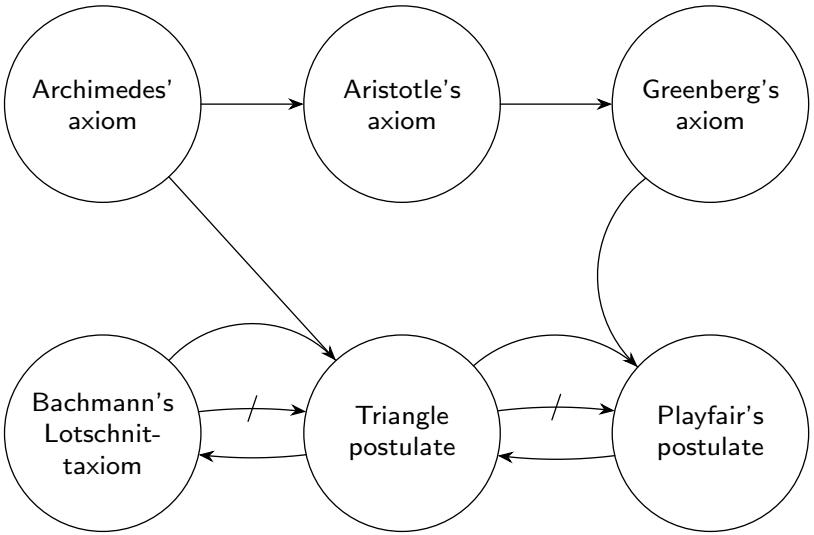
Identity for betweenness	$A\text{—}B\text{—}A \Rightarrow A = B$
Transitivity for congruence	$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A\text{—}B\text{—}E \wedge BE \equiv CD$
Pasch	$A\text{—}P\text{—}C \wedge B\text{—}Q\text{—}C \Rightarrow \exists X, P\text{—}X\text{—}B \wedge Q\text{—}X\text{—}A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A\text{—}B\text{—}C \wedge A'\text{—}B'\text{—}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A\text{—}B\text{—}C \wedge \neg B\text{—}C\text{—}A \wedge \neg C\text{—}A\text{—}B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \Rightarrow$ $A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B$
Euclid	$A\text{—}D\text{—}T \wedge B\text{—}D\text{—}C \wedge A \neq D \Rightarrow$ $\exists XY, A\text{—}B\text{—}X \wedge A\text{—}C\text{—}Y \wedge X\text{—}T\text{—}Y$
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, X \in \Xi \wedge Y \in \Upsilon \Rightarrow A\text{—}X\text{—}Y)) \Rightarrow$ $\exists B, (\forall XY, X \in \Xi \wedge Y \in \Upsilon \Rightarrow X\text{—}B\text{—}Y)$

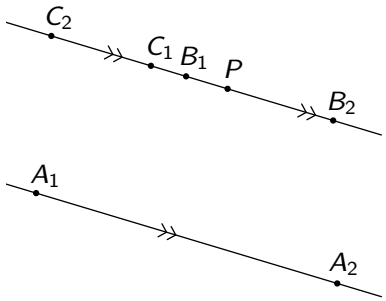
Parallel postulates are not *equivalent*

Parallel postulates are not *equivalent*

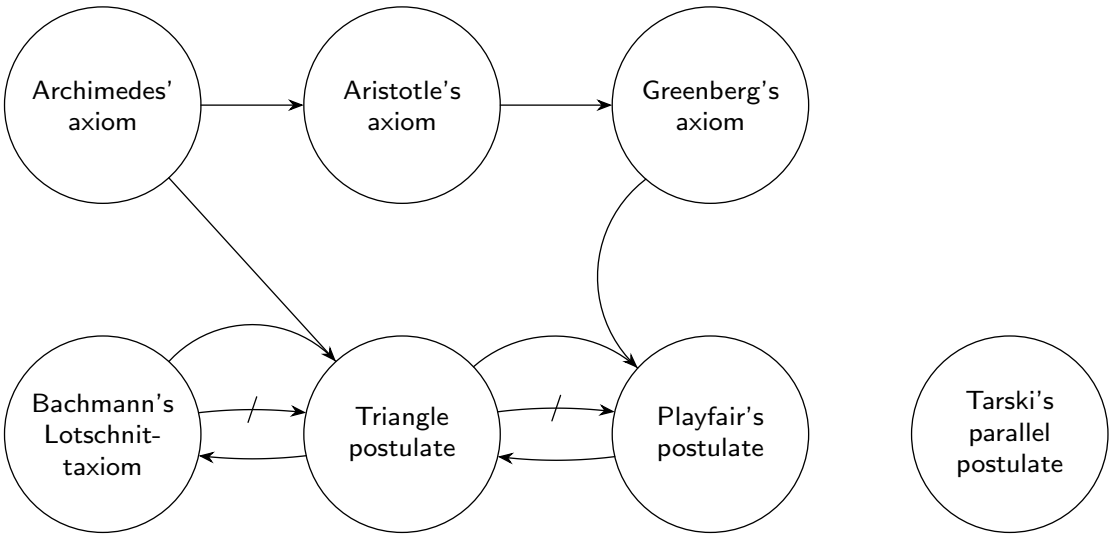


Parallel postulates are not *equivalent*

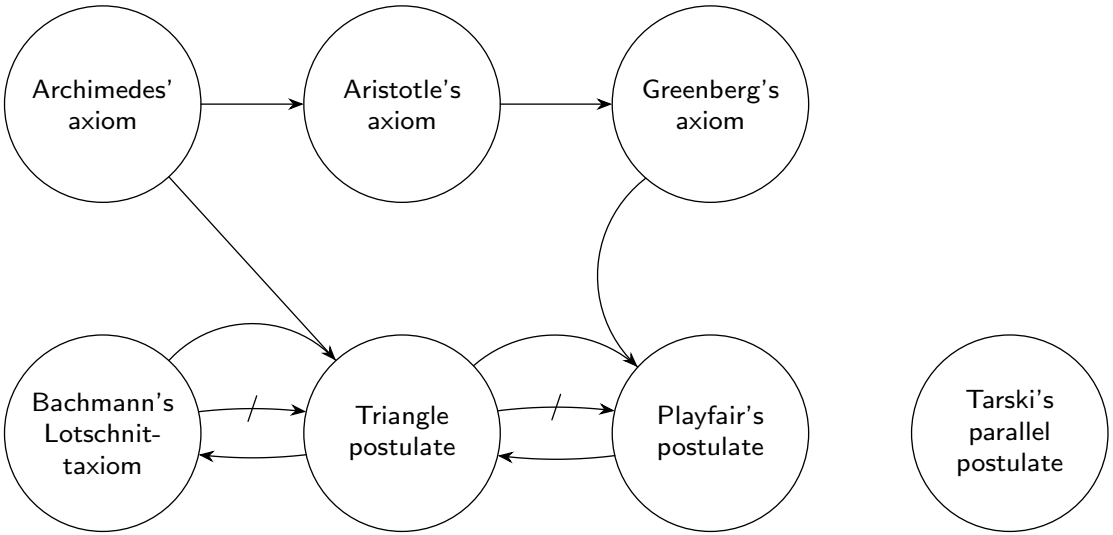


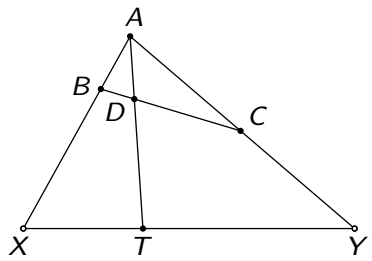


Parallel postulates are not *equivalent*

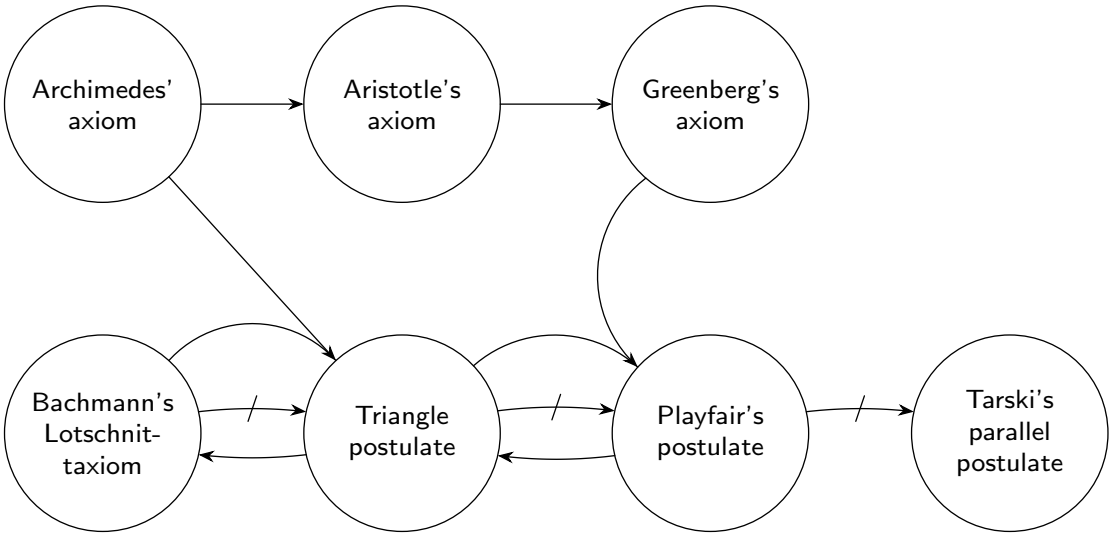


Parallel postulates are not *equivalent*

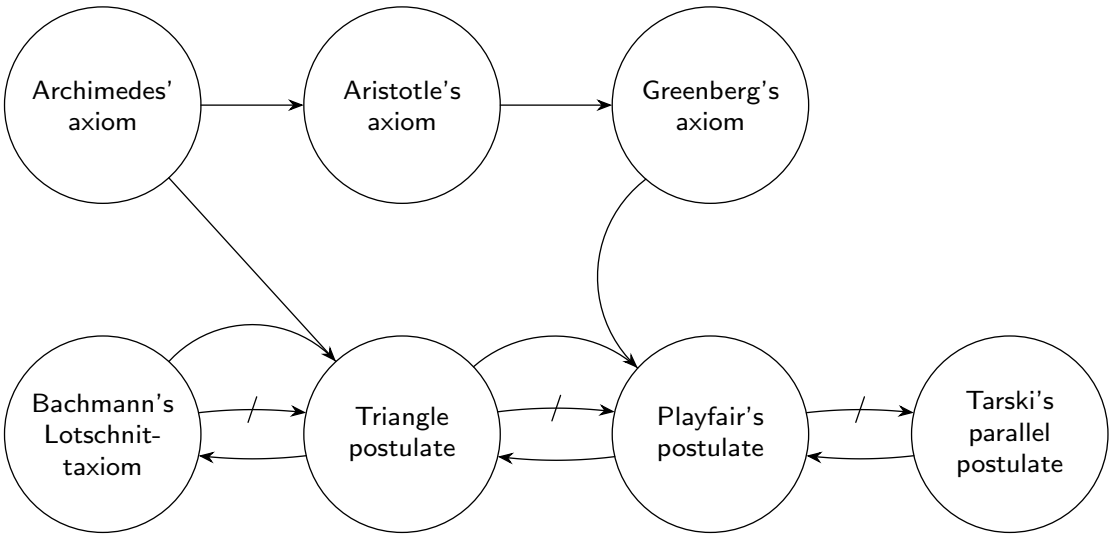




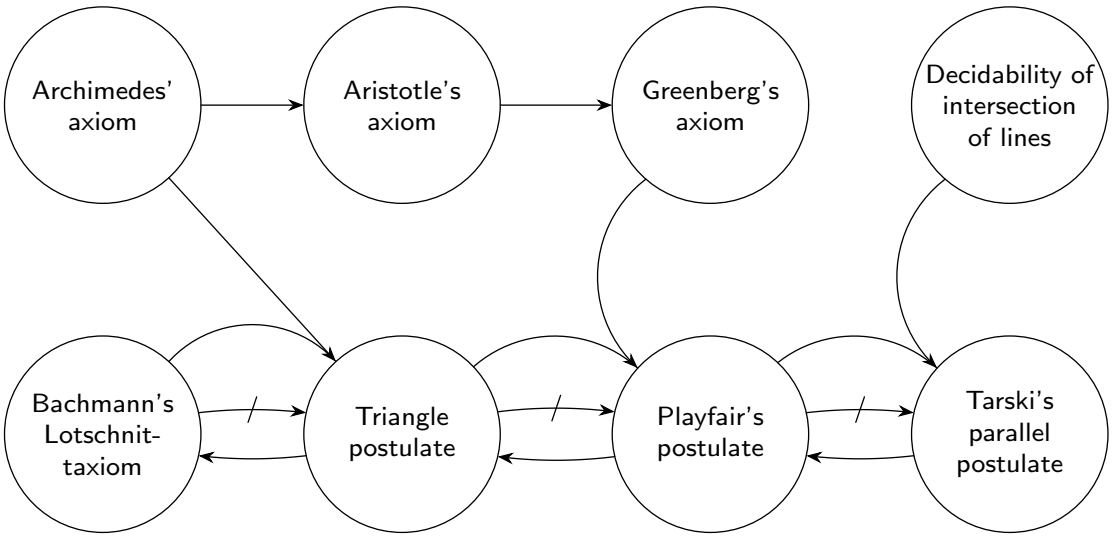
Parallel postulates are not *equivalent*



Parallel postulates are not *equivalent*

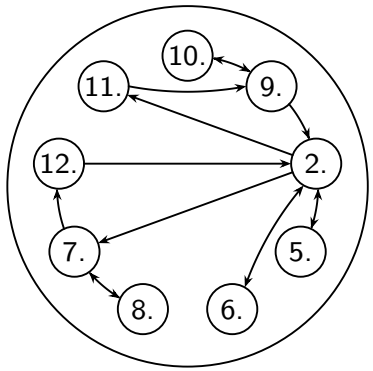


Parallel postulates are not *equivalent*

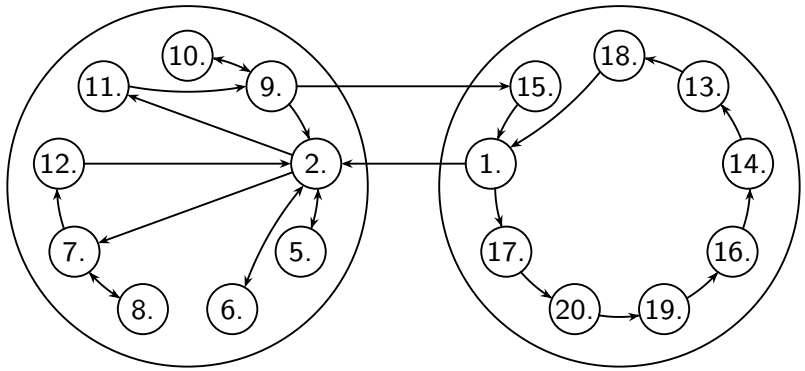


How to classify the postulates?

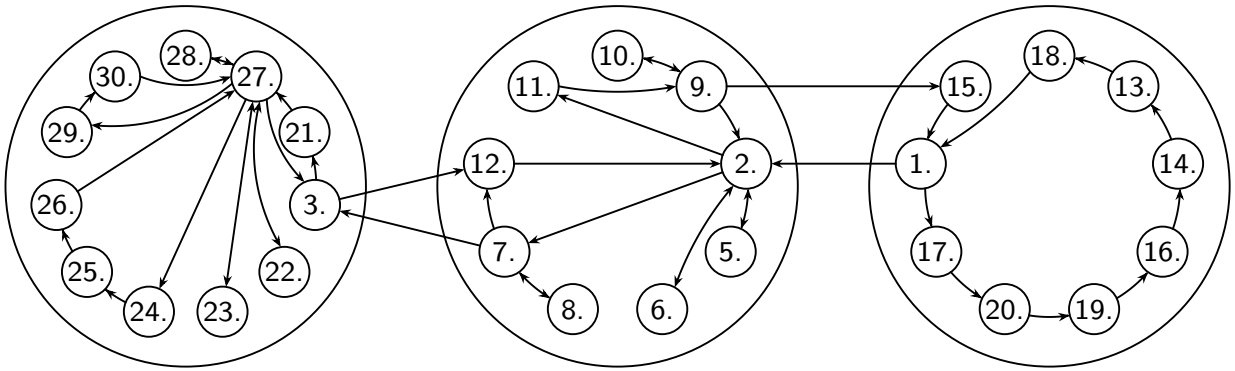
How to classify the postulates?



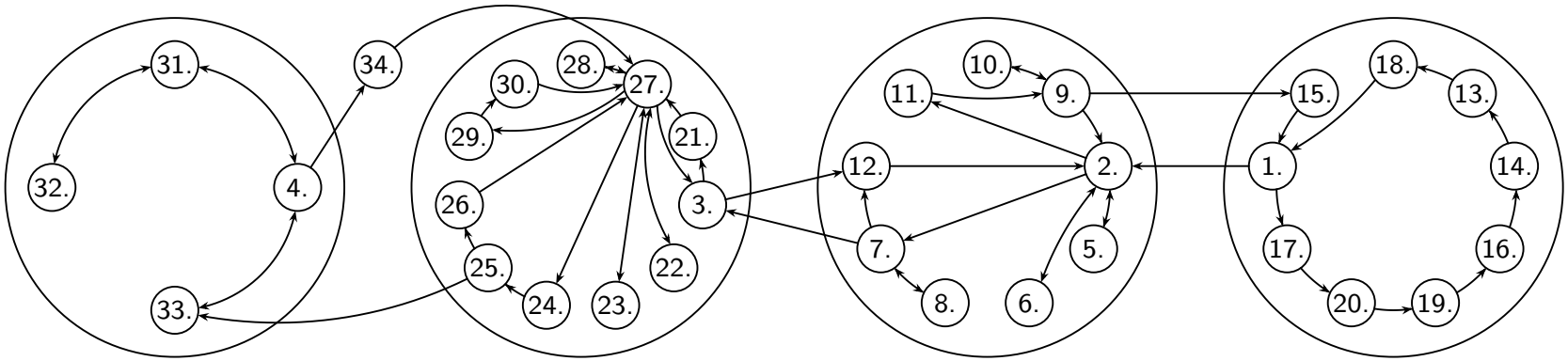
How to classify the postulates?



How to classify the postulates?



How to classify the postulates?



How to classify the postulates?

Pursuing the project faithfully will require that we take the extreme measure of shutting out the entreaties of our intuitions and imaginations - a forced separation of mental powers that will quite understandably be confusing and difficult to maintain [...].

(Richard J. Trudeau)

The axioms

The axioms

Transitivity for betweenness	$A—B—D \wedge B—C—D \Rightarrow A—B—C$
Transitivity for congruence	$AB \equiv EF \wedge CD \equiv EF \Rightarrow AB \equiv CD$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A—B—E \wedge BE \equiv CD$
Pasch	$A—X—Q \wedge C—Q—B \Rightarrow \exists P, A—P—C \wedge B—X—P$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A—B—C \wedge A'—B'—C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A—B—C \wedge \neg B—C—A \wedge \neg C—A—B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \wedge$ $A \neq B \wedge B \neq C \wedge A \neq C \Rightarrow$ $A—B—C \vee B—C—A \vee C—A—B$
Euclid	$A—D—T \wedge B—D—C \wedge A \neq D \Rightarrow$ $\exists XY, A—B—X \wedge A—C—Y \wedge X—T—Y$
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow A—X—Y)) \Rightarrow$ $\exists B, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow X—B—Y \vee X = B \vee B = Y)$

The axioms

Symmetry for betweenness	$A\text{—}B\text{—}C \Rightarrow C\text{—}B\text{—}A$
Transitivity for betweenness	$A\text{—}B\text{—}D \wedge B\text{—}C\text{—}D \Rightarrow A\text{—}B\text{—}C$
Transitivity for congruence	$AB \equiv EF \wedge CD \equiv EF \Rightarrow AB \equiv CD$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A\text{—}B\text{—}E \wedge BE \equiv CD$
Pasch	$A\text{—}X\text{—}Q \wedge C\text{—}Q\text{—}B \Rightarrow \exists P, A\text{—}P\text{—}C \wedge B\text{—}X\text{—}P$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A\text{—}B\text{—}C \wedge A'\text{—}B'\text{—}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A\text{—}B\text{—}C \wedge \neg B\text{—}C\text{—}A \wedge \neg C\text{—}A\text{—}B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \wedge$ $A \neq B \wedge B \neq C \wedge A \neq C \Rightarrow$ $A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B$
Euclid	$A\text{—}D\text{—}T \wedge B\text{—}D\text{—}C \wedge A \neq D \Rightarrow$ $\exists XY, A\text{—}B\text{—}X \wedge A\text{—}C\text{—}Y \wedge X\text{—}T\text{—}Y$
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow A\text{—}X\text{—}Y)) \Rightarrow$ $\exists B, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow X\text{—}B\text{—}Y \vee X = B \vee B = Y)$

The axioms

Symmetry for betweenness	$A—B—C \Rightarrow C—B—A$
Transitivity for betweenness	$A—B—D \wedge B—C—D \Rightarrow A—B—C$
Transitivity for congruence	$AB \equiv EF \wedge CD \equiv EF \Rightarrow AB \equiv CD$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A—B—E \wedge BE \equiv CD$
Pasch	$A—X—Q \wedge C—Q—B \Rightarrow \exists P, A—P—C \wedge B—X—P$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A—B—C \wedge A'—B'—C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A—B—C \wedge \neg B—C—A \wedge \neg C—A—B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \wedge$ $A \neq B \wedge B \neq C \wedge A \neq C \Rightarrow$ $A—B—C \vee B—C—A \vee C—A—B$
Euclid	$A—D—T \wedge B—D—C \wedge A \neq D \Rightarrow$ $\exists XY, A—B—X \wedge A—C—Y \wedge X—T—Y$
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow A—X—Y)) \Rightarrow$ $\exists B, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow X—B—Y \vee X = B \vee B = Y)$

The axioms

Symmetry for betweenness	$A—B—C \Rightarrow C—B—A$
Transitivity for betweenness	$A—B—D \wedge B—C—D \Rightarrow A—B—C$
Transitivity for congruence	$AB \equiv EF \wedge CD \equiv EF \Rightarrow AB \equiv CD$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A—B—E \wedge BE \equiv CD$
Pasch	$A—X—Q \wedge C—Q—B \Rightarrow \exists P, A—P—C \wedge B—X—P$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A—B—C \wedge A'—B'—C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A—B—C \wedge \neg B—C—A \wedge \neg C—A—B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \wedge$ $A \neq B \wedge B \neq C \wedge A \neq C \Rightarrow$ $A—B—C \vee B—C—A \vee C—A—B$
Euclid	$A—D—T \wedge B—D—C \wedge A \neq D \Rightarrow$ $\exists XY, A—B—X \wedge A—C—Y \wedge X—T—Y$
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow A—X—Y)) \Rightarrow$ $\exists B, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow X—B—Y \vee X = B \vee B = Y)$
Point equality decidability	$X = Y \vee X \neq Y$

The axioms

Symmetry for betweenness	$A—B—C \Rightarrow C—B—A$
Transitivity for betweenness	$A—B—D \wedge B—C—D \Rightarrow A—B—C$
Transitivity for congruence	$AB \equiv EF \wedge CD \equiv EF \Rightarrow AB \equiv CD$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A—B—E \wedge BE \equiv CD$
Pasch	$A—X—Q \wedge C—Q—B \Rightarrow \exists P, A—P—C \wedge B—X—P$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A—B—C \wedge A'—B'—C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A—B—C \wedge \neg B—C—A \wedge \neg C—A—B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \wedge$ $A \neq B \wedge B \neq C \wedge A \neq C \Rightarrow$ $A—B—C \vee B—C—A \vee C—A—B$
Euclid	$A—D—T \wedge B—D—C \wedge A \neq D \Rightarrow$ $\exists XY, A—B—X \wedge A—C—Y \wedge X—T—Y$
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow A—X—Y)) \Rightarrow$ $\exists B, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow X—B—Y \vee X = B \vee B = Y)$
Point equality decidability	$X = Y \vee X \neq Y$

The axioms

Symmetry for betweenness	$A\text{—}B\text{—}C \Rightarrow C\text{—}B\text{—}A$
Transitivity for betweenness	$A\text{—}B\text{—}D \wedge B\text{—}C\text{—}D \Rightarrow A\text{—}B\text{—}C$
Transitivity for congruence	$AB \equiv EF \wedge CD \equiv EF \Rightarrow AB \equiv CD$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A\text{—}B\text{—}E \wedge BE \equiv CD$
Pasch	$A\text{—}P\text{—}C \wedge B\text{—}Q\text{—}C \wedge A \neq P \wedge P \neq C \wedge$ $B \neq Q \wedge Q \neq C \wedge \neg(A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B) \Rightarrow$ $\exists X, P\text{—}X\text{—}B \wedge Q\text{—}X\text{—}A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A\text{—}B\text{—}C \wedge A'\text{—}B'\text{—}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A\text{—}B\text{—}C \wedge \neg B\text{—}C\text{—}A \wedge \neg C\text{—}A\text{—}B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \wedge$ $A \neq B \wedge B \neq C \wedge A \neq C \Rightarrow$ $A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B$
Euclid	$A\text{—}D\text{—}T \wedge B\text{—}D\text{—}C \wedge A \neq D \Rightarrow$ $\exists XY, A\text{—}B\text{—}X \wedge A\text{—}C\text{—}Y \wedge X\text{—}T\text{—}Y$
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow A\text{—}X\text{—}Y)) \Rightarrow$ $\exists B, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow X\text{—}B\text{—}Y \vee X = B \vee B = Y)$
Point equality decidability	$X = Y \vee X \neq Y$

The axioms

Symmetry for betweenness	$A—B—C \Rightarrow C—B—A$
Transitivity for betweenness	$A—B—D \wedge B—C—D \Rightarrow A—B—C$
Transitivity for congruence	$AB \equiv EF \wedge CD \equiv EF \Rightarrow AB \equiv CD$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A—B—E \wedge BE \equiv CD$
Pasch	$A—P—C \wedge B—Q—C \wedge A \neq P \wedge P \neq C \wedge$ $B \neq Q \wedge Q \neq C \wedge \neg(A—B—C \vee B—C—A \vee C—A—B) \Rightarrow$ $\exists X, P—X—B \wedge Q—X—A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A—B—C \wedge A'—B'—C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A—B—C \wedge \neg B—C—A \wedge \neg C—A—B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \wedge$ $A \neq B \wedge B \neq C \wedge A \neq C \Rightarrow$ $A—B—C \vee B—C—A \vee C—A—B$
Euclid	$A—D—T \wedge B—D—C \wedge A \neq D \Rightarrow$ $\exists XY, A—B—X \wedge A—C—Y \wedge X—T—Y$
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow A—X—Y)) \Rightarrow$ $\exists B, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow X—B—Y \vee X = B \vee B = Y)$
Point equality decidability	$X = Y \vee X \neq Y$

The axioms

Symmetry for betweenness	$A—B—C \Rightarrow C—B—A$
Transitivity for betweenness	$A—B—D \wedge B—C—D \Rightarrow A—B—C$
Transitivity for congruence	$AB \equiv EF \wedge CD \equiv EF \Rightarrow AB \equiv CD$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A—B—E \wedge BE \equiv CD$
Pasch	$A—P—C \wedge B—Q—C \wedge A \neq P \wedge P \neq C \wedge$ $B \neq Q \wedge Q \neq C \wedge \neg(A—B—C \vee B—C—A \vee C—A—B) \Rightarrow$ $\exists X, P—X—B \wedge Q—X—A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A—B—C \wedge A'—B'—C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A—B—C \wedge \neg B—C—A \wedge \neg C—A—B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \wedge$ $A \neq B \wedge B \neq C \wedge A \neq C \Rightarrow$ $A—B—C \vee B—C—A \vee C—A—B$
Proclus	
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow A—X—Y)) \Rightarrow$ $\exists B, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow X—B—Y \vee X = B \vee B = Y)$
Point equality decidability	$X = Y \vee X \neq Y$

A few definitions

A few definitions

Collinearity

$$A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B$$

A few definitions

Collinearity

$$A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B$$

Coplanarity

$$\exists X, (\mathrm{Col}\ A\ B\ X \wedge \mathrm{Col}\ C\ D\ X) \vee (\mathrm{Col}\ A\ C\ X \wedge \mathrm{Col}\ B\ D\ X) \vee (\mathrm{Col}\ A\ D\ X \wedge \mathrm{Col}\ B\ C\ X)$$

A few definitions

Collinearity

$$A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B$$

Coplanarity

$$\exists X, (\mathsf{Col}\ A\ B\ X \wedge \mathsf{Col}\ C\ D\ X) \vee (\mathsf{Col}\ A\ C\ X \wedge \mathsf{Col}\ B\ D\ X) \vee (\mathsf{Col}\ A\ D\ X \wedge \mathsf{Col}\ B\ C\ X)$$

Strict parallelism

$$A \neq B \wedge C \neq D \wedge \mathsf{Cp}\ A\ B\ C\ D \wedge \neg\exists X, \mathsf{Col}\ A\ B\ X \wedge \mathsf{Col}\ C\ D\ X$$

A few definitions

Collinearity

$$A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B$$

Coplanarity

$$\exists X, (\mathrm{Col}\ A\ B\ X \wedge \mathrm{Col}\ C\ D\ X) \vee (\mathrm{Col}\ A\ C\ X \wedge \mathrm{Col}\ B\ D\ X) \vee (\mathrm{Col}\ A\ D\ X \wedge \mathrm{Col}\ B\ C\ X)$$

Strict parallelism

$$A \neq B \wedge C \neq D \wedge \mathrm{Cp}\ A\ B\ C\ D \wedge \neg\exists X, \mathrm{Col}\ A\ B\ X \wedge \mathrm{Col}\ C\ D\ X$$

Parallelism

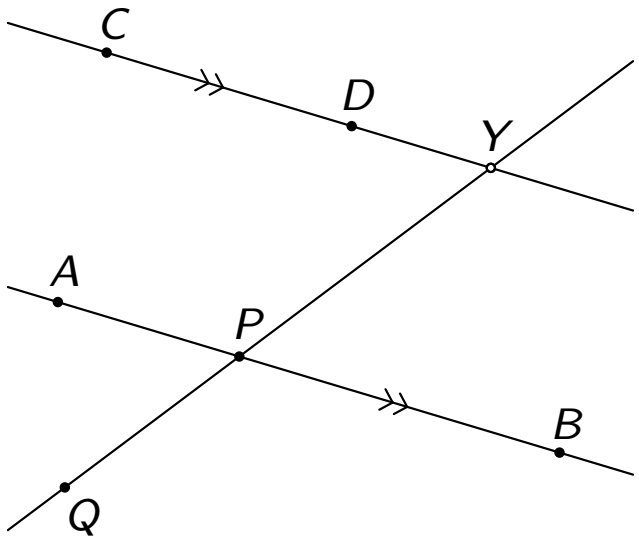
$$AB \parallel_s CD \vee (A \neq B \wedge C \neq D \wedge \mathrm{Col}\ A\ C\ D \wedge \mathrm{Col}\ B\ C\ D)$$

Proclus' axiom

Proclus' axiom

Axiom (Proclus' axiom)

$$AB \parallel CD \wedge \mathrm{Col}\ A\ B\ P \wedge \neg\, \mathrm{Col}\ A\ B\ Q \Rightarrow$$
$$\exists Y, \mathrm{Col}\ C\ D\ Y \wedge \mathrm{Col}\ P\ Q\ Y$$



The axioms

The axioms

Symmetry for betweenness	$A—B—C \Rightarrow C—B—A$
Transitivity for betweenness	$A—B—D \wedge B—C—D \Rightarrow A—B—C$
Transitivity for congruence	$AB \equiv EF \wedge CD \equiv EF \Rightarrow AB \equiv CD$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A—B—E \wedge BE \equiv CD$
Pasch	$A—P—C \wedge B—Q—C \wedge A \neq P \wedge P \neq C \wedge$ $B \neq Q \wedge Q \neq C \wedge \neg(A—B—C \vee B—C—A \vee C—A—B) \Rightarrow$ $\exists X, P—X—B \wedge Q—X—A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A—B—C \wedge A'—B'—C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A—B—C \wedge \neg B—C—A \wedge \neg C—A—B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \wedge$ $A \neq B \wedge B \neq C \wedge A \neq C \Rightarrow$ $A—B—C \vee B—C—A \vee C—A—B$
Proclus	
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow A—X—Y)) \Rightarrow$ $\exists B, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow X—B—Y \vee X = B \vee B = Y)$
Point equality decidability	$X = Y \vee X \neq Y$

The axioms

Symmetry for betweenness	$A—B—C \Rightarrow C—B—A$
Transitivity for betweenness	$A—B—D \wedge B—C—D \Rightarrow A—B—C$
Transitivity for congruence	$AB \equiv EF \wedge CD \equiv EF \Rightarrow AB \equiv CD$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A—B—E \wedge BE \equiv CD$
Pasch	$A—P—C \wedge B—Q—C \wedge A \neq P \wedge P \neq C \wedge$ $B \neq Q \wedge Q \neq C \wedge \neg(A—B—C \vee B—C—A \vee C—A—B) \Rightarrow$ $\exists X, P—X—B \wedge Q—X—A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A—B—C \wedge A'—B'—C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A—B—C \wedge \neg B—C—A \wedge \neg C—A—B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \wedge$ $A \neq B \wedge B \neq C \wedge A \neq C \Rightarrow$ $A—B—C \vee B—C—A \vee C—A—B$
Proclus	
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow A—X—Y)) \Rightarrow$ $\exists B, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow X—B—Y \vee X = B \vee B = Y)$
Point equality decidability	$X = Y \vee X \neq Y$

The axioms

Symmetry for betweenness	$A\text{—}B\text{—}C \Rightarrow C\text{—}B\text{—}A$
Transitivity for betweenness	$A\text{—}B\text{—}D \wedge B\text{—}C\text{—}D \Rightarrow A\text{—}B\text{—}C$
Transitivity for congruence	$AB \equiv EF \wedge CD \equiv EF \Rightarrow AB \equiv CD$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A\text{—}B\text{—}E \wedge BE \equiv CD$
Pasch	$A\text{—}P\text{—}C \wedge B\text{—}Q\text{—}C \wedge A \neq P \wedge P \neq C \wedge$ $B \neq Q \wedge Q \neq C \wedge \neg(A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B) \Rightarrow$ $\exists X, P\text{—}X\text{—}B \wedge Q\text{—}X\text{—}A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A\text{—}B\text{—}C \wedge A'\text{—}B'\text{—}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A\text{—}B\text{—}C \wedge \neg B\text{—}C\text{—}A \wedge \neg C\text{—}A\text{—}B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \wedge$ $A \neq B \wedge B \neq C \wedge A \neq C \Rightarrow$ $A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B$
Proclus	
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow A\text{—}X\text{—}Y)) \Rightarrow$ $\exists B, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow X\text{—}B\text{—}Y \vee X = B \vee B = Y)$
Point equality decidability	$X = Y \vee X \neq Y$

The axioms

Symmetry for betweenness	$A\text{—}B\text{—}C \Rightarrow C\text{—}B\text{—}A$
Transitivity for betweenness	$A\text{—}B\text{—}D \wedge B\text{—}C\text{—}D \Rightarrow A\text{—}B\text{—}C$
Transitivity for congruence	$AB \equiv EF \wedge CD \equiv EF \Rightarrow AB \equiv CD$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A\text{—}B\text{—}E \wedge BE \equiv CD$
Pasch	$A\text{—}P\text{—}C \wedge B\text{—}Q\text{—}C \wedge A \neq P \wedge P \neq C \wedge$ $B \neq Q \wedge Q \neq C \wedge \neg(A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B) \Rightarrow$ $\exists X, P\text{—}X\text{—}B \wedge Q\text{—}X\text{—}A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A\text{—}B\text{—}C \wedge A'\text{—}B'\text{—}C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A\text{—}B\text{—}C \wedge \neg B\text{—}C\text{—}A \wedge \neg C\text{—}A\text{—}B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \wedge$ $A \neq B \wedge B \neq C \wedge A \neq C \Rightarrow$ $A\text{—}B\text{—}C \vee B\text{—}C\text{—}A \vee C\text{—}A\text{—}B$
Proclus	
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow A\text{—}X\text{—}Y)) \Rightarrow$ $\exists B, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow X\text{—}B\text{—}Y \vee X = B \vee B = Y)$
Point equality decidability	$X = Y \vee X \neq Y$

The axioms

Symmetry for betweenness	$A—B—C \Rightarrow C—B—A$
Transitivity for betweenness	$A—B—D \wedge B—C—D \Rightarrow A—B—C$
Transitivity for congruence	$AB \equiv EF \wedge CD \equiv EF \Rightarrow AB \equiv CD$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A—B—E \wedge BE \equiv CD$
Pasch	$A—P—C \wedge B—Q—C \wedge A \neq P \wedge P \neq C \wedge$ $B \neq Q \wedge Q \neq C \wedge \neg(A—B—C \vee B—C—A \vee C—A—B) \Rightarrow$ $\exists X, P—X—B \wedge Q—X—A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A—B—C \wedge A'—B'—C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A—B—C \wedge \neg B—C—A \wedge \neg C—A—B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \wedge$ $A \neq B \wedge B \neq C \wedge A \neq C \Rightarrow$ $A—B—C \vee B—C—A \vee C—A—B$
Proclus	
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow A—X—Y)) \Rightarrow$ $\exists B, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow X—B—Y \vee X = B \vee B = Y)$
Point equality decidability	$X = Y \vee X \neq Y$

The axioms

Symmetry for betweenness	$A—B—C \Rightarrow C—B—A$
Transitivity for betweenness	$A—B—D \wedge B—C—D \Rightarrow A—B—C$
Transitivity for congruence	$AB \equiv EF \wedge CD \equiv EF \Rightarrow AB \equiv CD$
Reflexivity for congruence	$AB \equiv BA$
Identity for congruence	$AB \equiv CC \Rightarrow A = B$
Segment Construction	$\exists E, A—B—E \wedge BE \equiv CD$
Pasch	$A—P—C \wedge B—Q—C \wedge A \neq P \wedge P \neq C \wedge$ $B \neq Q \wedge Q \neq C \wedge \neg(A—B—C \vee B—C—A \vee C—A—B) \Rightarrow$ $\exists X, P—X—B \wedge Q—X—A$
Five-Segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A—B—C \wedge A'—B'—C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
Lower 2-Dimensional	$\exists ABC, \neg A—B—C \wedge \neg B—C—A \wedge \neg C—A—B$
Upper 2-Dimensional	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \wedge$ $A \neq B \wedge B \neq C \wedge A \neq C \Rightarrow$ $A—B—C \vee B—C—A \vee C—A—B$
Proclus	
Continuity	$\forall \Xi \Upsilon, (\exists A, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow A—X—Y)) \Rightarrow$ $\exists B, (\forall XY, \Xi X \wedge \Upsilon Y \Rightarrow X—B—Y \vee X = B \vee B = Y)$
Point equality decidability	$X = Y \vee X \neq Y$

Results

Results

- An independent version of Tarski's system of geometry.

Results

- An independent version of Tarski's system of geometry.
- Formalization of 10 out of 13 counter-models in Coq (with Stéphane Kastenbaum, Marius Hinge, Clément Saintier and Jihad Zraibi).

Results

- An independent version of Tarski's system of geometry.
- Formalization of 10 out of 13 counter-models in Coq (with Stéphane Kastenbaum, Marius Hinge, Clément Saintier and Jihad Zraibi).
- We had to correct one of Gupta's models.

Results

- An independent version of Tarski's system of geometry.
- Formalization of 10 out of 13 counter-models in Coq (with Stéphane Kastenbaum, Marius Hinge, Clément Saintier and Jihad Zraibi).
- We had to correct one of Gupta's models.
- Inner Pasch holds in the models that were given for outer Pasch.

Thank you!