# Local proof-theoretic foundations and proof-theoretic tameness in ordinary mathematics

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Covers numerous fixed point, zero-finding, minimization or equilibirium problems with iterative procedures $\boldsymbol{(x_n)}$ s.t. e.g. in the case of fixed point problems one has

$$\textbf{(1)}\ d(x_n, Tx_n) \stackrel{n\to\infty}{\to} = 0 \quad \text{or even}$$

**(2)** $\boldsymbol{(x_n)}$ strongly converges to the fixed point of $\boldsymbol{T}$.

For such situations, **special designed** (for **particular classes** of spaces $X$ and mappings $T$) **logical metatheorems** (K. TAMS 2005, Gerhardy/K. TAMS 2008) have been designed which guarantee the extractability of explicit uniform bounds for $\forall \underline{\boldsymbol{x}} \in \mathbb{N}, \mathbb{N}^{\mathbb{N}}, \boldsymbol{X}, \boldsymbol{X}^{\boldsymbol{X}}, \boldsymbol{X}^{\mathbb{N}} \ldots \exists \boldsymbol{n} \in \mathbb{N}\, \boldsymbol{A}(\underline{\boldsymbol{x}}, \boldsymbol{n})$-theorems.

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- **compact** metric spaces (**if separability is used**) and
- bounded subsets of **abstract** metric structures $X$.

# Formal systems for analysis with abstract metric spaces $X$

**Types:** (i) $\mathbb{N}, X$ are types, (ii) with $\rho, \tau$ also $\rho \to \tau$ is a type.

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$\mathcal{A}^{\omega}[X, \|\cdot\| \ldots]$ e.g. results by adding constants with axioms expressing that $(X, \|\cdot\|)$ is normed, uniformly convex, Hilbert.

# Keeping track of uniform bounds: majorization

$y, x$ functionals of types $\rho, \widehat{\rho} := \rho[\mathbb{N}/X]$ :

$$\boldsymbol{x}^{\mathbb{N}} \gtrsim_{\mathbb{N}} \boldsymbol{y}^{\mathbb{N}} :\equiv \boldsymbol{x} \geq \boldsymbol{y}$$
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**Example:**

$$f^* \gtrsim_{X \to X} f \equiv \forall n \in \mathbb{N}, x \in X[n \geq \|x\| \to f^*(n) \geq \|f(x)\|].$$

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$\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{X}$ **nonexpansive:** $\|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})\| \leq \|\boldsymbol{x} - \boldsymbol{y}\|$.

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In a **metric setting: reference point** $a \in X : \boldsymbol{x} \geq \boldsymbol{d}(\boldsymbol{a}, \boldsymbol{y})$.

Special case of **general logical metatheorems** ($T$ nonexpansive):

## Corollary (Gerhardy/K., TAMS 2008)

If $\mathcal{A}^\omega[X, \|\cdot\|]$ proves (***K*** represented compact metric space)

$\forall \boldsymbol{x} \in \mathbb{N}^{\mathbb{N}}\, \forall \boldsymbol{y} \in \boldsymbol{K}\, \forall \boldsymbol{z} \in \boldsymbol{X}\, \forall \boldsymbol{T} : \boldsymbol{X} \to \boldsymbol{X}\, \big(\boldsymbol{T} \text{ n.e.} \to \exists \boldsymbol{v} \in \mathbb{N}\, \boldsymbol{A}_{\exists}\big),$

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then one can extract a **computable functional** $\Phi : \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$

s.t. in all normed spaces $\boldsymbol{X}$ it holds that

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**Method:** Novel forms of Gödel's functional interpretation!

## Applicability of Metatheorem

- Applied to asymptotic regularity statements $d(x_n, Tx_n) \to 0$, the corollary often gives full rates of convergence, e.g. because $(d(x_n, Tx_n))$ is nonincreasing so that $d(x_n, Tx_n) \to 0 \in \forall\exists$.

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- From proofs of the convergence of $(x_n)$ itself, one may only get **rates of metastability** $\Phi$ (Kreisel 1951, K.05, Tao 07) s.t.

$$\forall k \in \mathbb{N}\, \forall g \in \mathbb{N}^{\mathbb{N}} \exists n \in \mathbb{N} \forall i, j \in [n, n+g(n)]\, (d(x_i, x_j) < 2^{-k}) \in \forall\exists.$$

- **Admissible abstract structures:** metric, hyperbolic, CAT(0), CAT($\kappa > 0$), Ptolemy, normed, their completions, Hilbert, uniformly convex, uniformly smooth (not: separable, strictly convex or smooth) spaces, abstract $L^p$- and $C(K)$-spaces (and all other normed structures axiomatizable in positive bounded logic (in the sense of Henson, Iovino, Ben-Yaacov etc.).

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- Uses of **ultraproducts** made in model theory can often be replaced by a **proof-theoretic uniform boundedness principle UB** which can be **eliminated** from proofs without contributing to the extracted bounds (K. ENTCS 2006, Engracia 2009, Günzel/K. Adv. Math. 2016). Recently **UB** has been used to replace sequential weak compactness (Ferreira, Leuştean, Pinto, Adv. Math. to appear).

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- Except for 2 cases, all **rates of metastability** are of essentially the form

$$\Phi(\underline{a}, g) = (\chi_1(\underline{a}) \circ g \circ \chi_2(\underline{a}))^{B(\underline{a})}(0)$$

for simple (essentially polynomial) functions $\chi_1, \chi_2, B$ in majorants $\underline{a}$ of the parameters of the problem.

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Implies: **algorithmic learnability** of a rate of convergence which - if a gap condition is satisfied - yields **oscillation bounds** (K./Safarik APAL 2014, Avigad/Rute ETDS 2015).

## Proof-theoretic versus model-theoretic tameness

- In the recent book 'Model Theory and the Philosophy of Mathematical Practice: Formalization without Foundationalism', John Baldwin has argued that model theory became successful in applications to core mathematics by focusing on **local** foundations/formalizations rather than **global** ones and on **tame structures** (e.g. o-minimal ones).

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- We argue, that in a related way, also ‘proof mining’ is successful by focusing on **specific classes of problems** (e.g. iterations of nonlinear operators $T : C \to C$ on general convex subsets of abstract classes of normed or geodesic spaces).

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- In contrast to model-theoretic tameness, quantification over $\mathbb{N}$ and inductions etc. are crucially used in connection with convergence statements so that **Gödel-phenomena could occur in principle**.
- A different form of **'proof-theoretic tameness'** in existing ordinary (nonlinear) analysis largely leads **to extractable bounds of very low complexity**.
- **Geometric properties** such as uniform convexity and smoothness etc. **more important than complicated inductions**.

# Proof-theoretic versus model-theoretic tameness

- To **detect** **proof-theoretic tameness** **requires** to actually carry out the **proof analysis** (though usually some rough upper bound on the complexity can be obtained from proof-theoretic conservation results).

## Proof-theoretic tameness in practice I:

## Polynomial rate of asymptotic regularity in Bauschke's solution of the 'zero displacement conjecture'

Consider a Hilbert space $\boldsymbol{H}$ and nonempty closed and convex subsets $\boldsymbol{C_1, \ldots, C_N \subseteq H}$ with metric projections $\boldsymbol{P_{C_i}}$, define $\boldsymbol{T := P_{C_N} \circ \ldots \circ P_{C_1}}$. In 2003 Bauschke proved the 'zero displacement conjecture':

$$\boldsymbol{\|T^{n+1}x - T^n x\| \to 0 \quad (x \in H).}$$

Previously only known for $N = 2$ or $Fix(T) \neq \emptyset$ (or even $\bigcap_{i=1}^N C_i \neq \emptyset$) or $C_i$ half spaces etc. starting with von Neumann.

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Previously only known for $N = 2$ or $Fix(T) \neq \emptyset$ (or even $\bigcap_{i=1}^N C_i \neq \emptyset$) or $C_i$ half spaces etc. starting with von Neumann. Proof uses abstract theory of maximal monotone operators: Minty's theorem, Brézis-Haraux theorem, Rockafellar's maximal monotonicity and sum theorems, strongly nonexpansive mappings, conjugate functions, normal cone operator...).

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Logical metatheorems, therefore, guarantee the extractability a uniform rate of asymptotic regularity which only depends on the error $\varepsilon > 0$, $N \in \mathbb{N}$ and **majorants** for $x \in H$ and $P_{C_1}, \ldots, P_{C_N}$: $b \geq \|x\|$ and $K \geq \|c_1\|, \ldots, \|c_N\|$ for some **arbitrary** points $c_1 \in C_1, \ldots, c_N \in C_N$:

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$$\|P_{C_i} 0\| \leq \|c_i\| \leq K.$$

Since the mappings $P_{C_i}$ are nonexpansive, the corollary guarantees a computable $\Phi(\varepsilon, N, b, K)$ s.t. for $b \geq \|x\|$

$$\forall \varepsilon > 0\, \forall n \geq \Phi(\varepsilon, N, b, K)\ (\|T^{n+1}x - T^n x\| < \varepsilon).$$

## Theorem (K. FoCM 2019)

$$\Phi(\varepsilon, N, b, K) := \left\lceil \frac{18b + 12\alpha(\varepsilon/6))}{\varepsilon} - 1 \right\rceil \left\lceil \left( \frac{D}{\omega(D, \tilde{\varepsilon})} \right) \right\rceil$$

is a **rate of asymptotic regularity** in Bauschke's result, where

$$\tilde{\varepsilon} := \frac{\varepsilon^2}{27b + 18\alpha(\varepsilon/6)}, D := 2b{+}NK, \omega(D, \tilde{\varepsilon}) := \frac{1}{16D}(\tilde{\varepsilon}/N)^2.$$

$$\alpha(\varepsilon) := \frac{(K^2 + N^3(N-1)^2K^2)N^2}{\varepsilon}.$$

Here $b \geq \|x\|$ and $K \geq \left(\sum_{i=1}^N \|c_i\|^2\right)^{\frac{1}{2}}$ for some $(c_1, \ldots, c_N) \in C_1 \times \ldots \times C_N$.

## Proof-theoretic tameness in practice II: Pursuit-evasion games: Lion-Man

Let $(X, d)$ be a uniquely geodesic space, $D > 0$. $L_0, M_0 \in A$ starting points of the lion $L$ and the man $M$. After $n$-steps, $M$ moves to any point $M_n$ s.t. $d(M_n, M_{n+1}) \leq D$ and $L$ moves via the geodesic $[L_n, M_n]$ s.t. $d(L_n, L_{n+1}) = \min\{D, d(L_n, M_n)\}$.

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Lopéz-Acedo/Nicolae/Piatek, Geom.Dedicat. to appear: if $X$ is a compact uniquely geodesic space with the betweenness property, then **the lion wins** i.e. $\mathbf{\lim d(L_{n+1}, M_n) = 0}$ (proof makes iterated use of sequential compactness, i.e. arithmetic comprehension **ACA**).

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'$\boldsymbol{\lim d(L_{n+1}, M_n) = 0}$' $\in \Pi^0_2$ since the sequence is nonincreasing!

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- Proof mining provides an explicit **rate of convergence** which only depends on $\Theta$ (in addition to $b \geq \mathrm{diam}(A), D, \varepsilon > 0$).
- **Moduli of uniform betweenness** can be **extracted** from **proofs of mere betweenness** for the admissible structures.

# Betweenness and uniform betweenness in metric spaces

**Definition (Diminnie and White 1981)**

Let $(X, d)$ be a metric space. $X$ satisfies the betweenness property if for any distinct points $x, y, z, w \in X$

$$\left.\begin{array}{l} d(x,y) + d(y,z) \leq d(x,z) \\ d(y,z) + d(z,w) \leq d(y,w) \end{array}\right\} \Rightarrow \; d(x,z) + d(z,w) \leq d(x,w).$$

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For normed spaces, betweenness follows from (but is strictly weaker than) strict convexity. It fails for $(\mathbb{R}^2, \|\cdot\|_\infty), (\mathbb{R}^2, \|\cdot\|_1)$ but holds for some nonstrictly convex spaces.

The functional interpretation upgrades betweenness to (equivalent in the compact case!):

**Definition (K., Lopéz-Acedo, Nicolae 2019)**

A metric space $(X, d)$ satisfies the uniform betweenness property with modulus $\Theta : (0,\infty)^3 \to (0,\infty)$ if

$$\forall \varepsilon, a, b > 0\, \forall x, y, z, w \in X$$
$$\left( \left\{ \begin{array}{l} \mathrm{sep}\{x,y,z,w\} \geq a \wedge \mathrm{diam}\{x,y,z,w\} \leq b \\ d(x,y) + d(y,z) \leq d(x,z) + \Theta(\varepsilon, a, b) \\ d(y,z) + d(z,w) \leq d(y,w) + \Theta(\varepsilon, a, b) \end{array} \right\} \\ \Rightarrow d(x,z) + d(z,w) \leq d(x,w) + \varepsilon \right).$$

**Definition (Lion-Man Game in general metric spaces)**

$X$ metric space, $D > 0$, $(M_n), (L_n)$ be sequences in $X$ s.t.

$$d(M_n, M_{n+1}) \leq D, \ d(L_{n+1}, L_n) + d(L_{n+1}, M_n) = d(L_n, M_n),$$

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Then $\langle (M_n), (L_n) \rangle$ is a **Lion-Man game** with speed $D > 0$.

Let $X$ be a **$b$-bounded** metric space with the uniform betweenness property with modulus $\Theta$ satisfying

$$\Theta(\varepsilon) := \Theta(\varepsilon, \varepsilon, b) \leq \varepsilon \quad \text{for all } \varepsilon > 0.$$

For $D > 0$ let $N \in \mathbb{N}$ be s.t. $b + 1 < ND$.

**Theorem (K./Lopéz-Acedo/Nicolae 2019)**

Let $X$ be a bounded metric space with the uniform betweenness property and $\langle (M_n), (L_n) \rangle$ be a Lion-Man game, speed $D > 0$. Then the Lion approaches the man arbitrarily close.

## Theorem (K./Lopéz-Acedo/Nicolae 2019)

Let $X$ be a bounded metric space with the uniform betweenness property and $\langle (M_n), (L_n) \rangle$ be a Lion-Man game, speed $D > 0$. Then the Lion approaches the man arbitrarily close.

Moreover with $b \geq \mathrm{diam}(X)$, $\Theta$, $N$ as above:

$$\forall \varepsilon > 0 \, \forall n \geq \Omega_{D,b,\Theta}(\varepsilon) \; (d(L_{n+1}, M_n) < \varepsilon),$$

where

$$\Omega_{D,b,\Theta}(\varepsilon) = N + N \left\lceil \frac{b}{\Theta^{(N)}(\alpha)} \right\rceil$$

with

$$0 < \alpha \leq \min\left\{ \frac{1}{N}, \frac{D}{2}, \frac{\varepsilon}{2} \right\}.$$

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For $L^p$:

$$\Phi(\varepsilon, b) = \begin{cases} \dfrac{p-1}{8}\dfrac{\varepsilon^2}{(b+\varepsilon)}, & \text{if } 1 < p \leq 2, \\ \dfrac{1}{p2^p}\dfrac{\varepsilon^p}{(b+\varepsilon)^{p-1}}, & \text{if } 2 < p < \infty. \end{cases}$$

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For **CAT**$(\kappa)$**-spaces** $X$, $\kappa > 0$, with $\mathrm{diam}(X) < \pi/(2\sqrt{\kappa})$:

$$\Phi(\varepsilon, b) = \frac{c}{16}\frac{\varepsilon^2}{b+\varepsilon}, \quad \text{where}$$

$c = (\pi - 2\sqrt{\kappa}\,\beta)\tan(\sqrt{\kappa}\,\beta)$ for any $0 < \beta \leq \pi/(2\sqrt{\kappa}) - \mathrm{diam}(X)$.

# Ptolemy spaces

**Definition**

A metric space $(X, d)$ is **Ptolemy** if for all $x, y, z, w \in X$

$$d(x,z)d(y,w) \leq d(x,y)d(z,w) + d(x,w)d(y,z).$$

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There are complete bounded Ptolemy spaces which are geodesic but **not uniquely geodesic**.

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## Proposition (Foertsch, Lytchak, Schroeder 2007)

There are complete bounded Ptolemy spaces which are geodesic but **not uniquely geodesic**.

## Proposition (K., Lopéz-Acedo, Nicolae 2019)

Let $(X, d)$ be a Ptolemy space. Then $\Theta(\varepsilon, a, b) := \sqrt{b^2 + \varepsilon a} - b$ is a modulus for the uniform betweenness property.

# A borderline case for proof-theoretic tameness

U. Kohlenbach, A. Sipoș, The finitary content of sunny nonexpansive retractions. arXiv:1812.04940 [math.FA], 2018.

## The Browder-Halpern result

Let $C \subseteq H$ be a bounded, closed and convex subset of a Hilbert space $H$. $T : C \to C$ be nonexpansive, $x_0 \in C$ and $t \in [0, 1)$.

$$\boldsymbol{T_t : C \to C, \quad T_t(x) := tTx + (1-t)x_0}$$

is a $t$-contraction and so has a unique point $x_t$ with $x_t = T_t x_t$.

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**Theorem (Browder 1967; Halpern 1967)**

$p := \lim_{t\to 1} x_t$ exists strongly and $p = P_{Fix(T)}x$, where $P$ denotes the metric projection.

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**Theorem (Browder 1967; Halpern 1967)**

$p := \lim_{t \to 1} x_t$ exists strongly and $p = P_{Fix(T)}x$, where $P$ denotes the metric projection.

Even in simple cases on $[0, 1]$ there is in general **no computable rate convergence**. However, a **primitive recursive in the simple form as mentioned above rate of metastability** is extracted in (K., Adv. Math. 2011).

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In the framework above, if $X$ is a **uniformly smooth Banach space**, then for all $x \in C$ we have that $\lim_{t \to 1} x_t := p$ exists and it is a fixed point of $T$. Moreover $p = Q_{Fix(T)}x$, where $Q$ is the unique **sunny nonexpansive retraction** onto $Fix(T)$.

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The convergence of numerous iterative algorithms in nonlinear analysis is based on Reich's theorem!

## Sunny nonexpansive retractions

Let $E$ be a nonempty subset of $C$ and $Q : C \to E$. We call $Q$ a **retraction** if for all $x \in E$, $Qx = x$. If $Q$ is a retraction, we call it **sunny** if for all $x \in C$ and $t \geq 0$, $Q(Qx + t(x - Qx)) = Qx$.

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**Proposition (Variational Inequality)**

A retraction $Q : C \to E$ is sunny and nonexpansive iff for all $x \in C$ and $y \in E$,

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The existence of (sunny) nonexpansive retractions onto $Fix(T)$ was first shown by R. Bruck in 1971,1973 using Zorn's lemma.

## One key step in the proof

Consider $f : C \to \mathbb{R}_+$ with $f(z) := \limsup_{n\to\infty} \|x_n - z\|$. Let $K$ be the set of minimizers of $f$. Claim: $K \cap Fix(T) \neq \emptyset$.

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Since $f$ is convex and continuous, $C$ is closed convex bounded nonempty, and $X$ is uniformly smooth, hence reflexive, we have that $K \neq \emptyset$.

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$$\begin{aligned} f(Ty) = \limsup_{n\to\infty} \|x_n - Ty\| &\le \limsup_{n\to\infty}(\|x_n - Tx_n\| + \|Tx_n - Ty\|) \\ &\le \limsup_{n\to\infty}(\|x_n - Tx_n\| + \|x_n - y\|) \\ &\le \limsup_{n\to\infty} \|x_n - Tx_n\| + \limsup_{n\to\infty} \|x_n - y\| \\ &= f(y) \le f(z), \end{aligned}$$

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so $Ty \in K$. Since $K$ is a closed convex bounded nonempty $T$-invariant subset of a uniformly smooth space, there is a $p \in K \cap Fix(T)$.

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While the functional interpretation of the $\varepsilon$-minimization in general requires Gödel primitive recursion of type 2 ($\boldsymbol{T_2}$) the actual extraction constructs a **rate of metastability** in $\boldsymbol{\Phi} \in \boldsymbol{T_1}$ (i.e. of Ackermann type).

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It may well be that a closer analysis of of $\Phi$ shows that it is already definable in $\boldsymbol{T_0}$ (in line with a classical result of Parsons that certain forms of type-1 primitive recursion can be reduced to $\boldsymbol{T_0}$).

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- Proofs which use highly abstract ‘ideal’ principles to prove concrete numerically meaningful results are most promising.
- Built suitable local proof-theoretic methods to cover such classes of proofs appropriately.

# Future of Proof Mining

- Find new 'organic connections' (Kreisel) between core mathematics and proof theory.
- New classes of results rewarding for proof-theoretic analysis.
- Proofs which use highly abstract 'ideal' principles to prove concrete numerically meaningful results are most promising.
- Built suitable local proof-theoretic methods to cover such classes of proofs appropriately.
- The area of analysis has been particularly fruitful. But other promising areas: geometry, algebra (see Simmons/Towsner Adv.Math.).

**Recent Surveys:**

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U. Kohlenbach, Proof-Theoretic Methods in Nonlinear Analysis. In. Proc. ICM 2018, Proc. ICM 2018, B. Sirakov, P. Ney de Souza, M. Viana (eds.), Vol. 2, pp. 61-82. World Scientific 2019.

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U. Kohlenbach, Local Formalizations in Nonlinear Analysis and Related Areas and Proof-Theoretic Tameness. To appear in forthcoming volume (eds. P. Weingartner, H.-P. Leeb) 'Kreisel's Interests - On the Foundations of Logic and Mathematics'.