Zero-one laws for provability logic and some of its siblings

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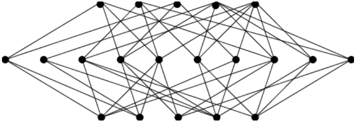
Logic Colloquium, Prague, 11-16 August, 2019



Overview

Zero-one laws: An introduction

Zero-one laws for modal logics

The three logics

Zero-one laws over relevant classes of finite models

How about zero-one laws for classes of finite frames?

Conclusions and current work

Introduction: First-order logic obeys a zero-one law

Let L be a language of first-order logic with $=$, but without function symbols (including constants).

Let $A_n(L)$ be the set of all (labelled) L-models with universe $\{1, \ldots, n\}$.

Let $\mu_n(\sigma)$ be the fraction of members of $A_n(L)$ in which σ is true: $\mu_n(\sigma) = \frac{|M \in A_n(L) : M \models \sigma|}{|A_n(L)|}$

Then for every $\sigma \in L$, $\lim_{n\to\infty} \mu_n(\sigma) = 1$ or $\lim_{n\to\infty} \mu_n(\varphi) = 0$.

That is, every formula is either almost surely true or almost surely false in finite models: a zero-one law.

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Example

$\forall x R(x, x)$ is almost surely false.
$\forall x \exists y R(x, y)$ is almost surely true.

A language of FOL with function symbol does not obey a zero-one law

Note that the condition that L does not contain function symbols (including constants) is necessary for the zero-one law to hold.

Example (No 0-1 law for language with unary function symbol)

Let L be $\{f\}$, and let $\sigma := \forall x \neg(f(x) = x)$.
Then for all n, $\mu_n(\sigma) = (\frac{n-1}{n})^n$ (values of f fixed independently)
So $\lim_{n\to\infty} \mu_n(\sigma) = \lim_{n\to\infty}(\frac{n-1}{n})^n = \frac{1}{e}$

History of the the zero-one law

Glebskii, Kogan, Liogon'kii and Talanov (1969) first proved the zero-one law for first-order logic.

It was also proved later but independently by Fagin (1976).

Ronald Fagin

Carnap (1950) had already proved the zero-one law for first-order languages with only unary predicate symbols.

Sketch of Fagin's proof for $L = \{R\}$: extension axioms

Fagin axiomatized the almost surely true formulas.
For example, let $L = \{R\}$, with R a binary predicate symbol.

Extension axioms (Gaifman 1964)

Let T contain all *extension axioms* of the form:

$$\forall x_1 \dots \forall x_k (\bigwedge_{i \neq j} x_i \neq x_j \rightarrow \exists y (\bigwedge_i y \neq x_i \wedge [\neg] R x_1 y \wedge \dots \wedge [\neg] R x_k y \wedge \\ [\neg] R y x_1 \wedge \dots \wedge [\neg] R x_k y \wedge \\ [\neg] R y y))$$

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Isomorphism lemma

If $\mathcal{M}_1$ and $\mathcal{M}_2$ are countably infinite models with $\mathcal{M}_1 \models T$ and $\mathcal{M}_2 \models T$, then $\mathcal{M}_1 \cong \mathcal{M}_2$.

Proof sketch: enumerate elements of $\mathcal{M}_1$ as $\{a_1, a_2, \ldots\}$ and those of $\mathcal{M}_2$ as $\{b_1, b_2, \ldots\}$, and do a back-and-forth construction.

Sketch of Fagin's proof for $L = \{R\}$: equivalences

Let $\mathcal{R}$ be the unique countably infinite model with $\mathcal{R} \models T$.

Theorem: Equivalences

For each formula σ, the following are equivalent:

1. $\mathcal{R} \models \sigma$
2. $T \vdash \sigma$
3. $\lim_{n\to\infty} \mu_n(\sigma) = 1$

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Proof sketch

$1 \Rightarrow 2$ Suppose $T \nvdash \sigma$, then there is a countable model $\mathcal{M} \models T + \neg\sigma$. By the Lemma, $\mathcal{M} \cong \mathcal{R}$, so $\mathcal{R} \not\models \sigma$

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$3 \Rightarrow 1$ Suppose $\mathcal{R} \not\models \sigma$, then $\mathcal{R} \models \neg\sigma$, so by $1 \Rightarrow 3$, $\lim_{n\to\infty} \mu_n(\neg\sigma) = 1$, so $\lim_{n\to\infty} \mu_n(\sigma) \neq 1$

Corollary: true in the random graph $\mathcal{R} \Leftrightarrow$ almost surely true

If $\mathcal{R} \models \sigma$, then σ is almost surely true; otherwise,
if $\mathcal{R} \not\models \sigma$, then σ is almost surely false

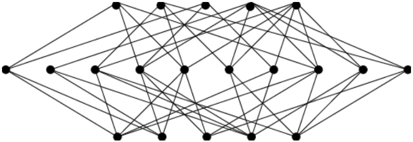
A surprising combinatorial result on finite partial orders I

Kleitman and Rothschild (1975) proved that with asymptotic probability 1, finite partial orders have a special structure. They can be divided into three levels:

- L_1, the set of minimal elements;
- L_2, the set of elements immediately succeeding elements in L_1;
- L_3, the set of elements immediately succeeding elements in L_2.

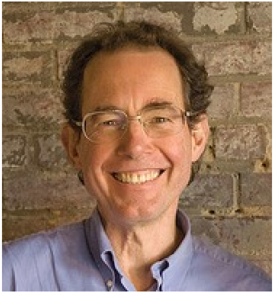
In partial orders of size n, the sizes of L_1 and L_3 both tend to $\frac{n}{4}$; the size of L_2 tends to $\frac{n}{2}$.

As n increases, each element in L_1 has as immediate successors asymptotically half of the elements of L_2;
and and each element in L_3 has as immediate predecessors asymptotically half of the elements of L_2.



A surprising combinatorial result on finite partial orders II

Kleitman and Rothschild's (1975) theorem holds for both non-strict (reflexive) and strict (irreflexive) partial orders.



Compton (1988) used this result to show that the zero-one law for first-order logic also holds with respect to partial orders.

Sketch of Compton's proof: extension axioms

Let $L = \{\leq\}$. Let T_{po} contain the usual axioms for partial orders, plus: $\forall x_0, x_1, x_2, x_3 (\bigwedge_{i \leq 2} x_i \leq x_{i+1} \rightarrow \bigvee_{i \leq 2} x_i = x_{i+1})$, plus:

Extension axioms

Levels L_1, L_2, L_3 are FO definable. Extension axioms:

- For all distinct $x_0, \ldots, x_{k-1}$ and $y_0, \ldots, y_{j-1}$ in L_2 and all distinct $z_0, \ldots, z_{l-1}$ in L_1, there is an element z in L_1 not equal to $z_0, \ldots, z_{l-1}$ such that: $\bigwedge_{i<k} z \leq x_i \ \wedge \bigwedge_{i<j} z \not\leq y_i$

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- For all distinct $x_0, \ldots, x_{k-1}$ and $y_0, \ldots, y_{j-1}$ in L_1 and all distinct $x'_0, \ldots, x'_{k'-1}$ and $y'_0, \ldots, y'_{j'-1}$ in L_3, and all distinct $z_0, \ldots, z_{l-1}$ in L_2, there is an element z in L_2 not equal to $z_0, \ldots, z_{l-1}$ such that:
  $\bigwedge_{i<k} x_i \leq z \; \wedge \bigwedge_{i<j} y_i \not\leq z \; \wedge \bigwedge_{i<k'} z \leq x'_i \; \wedge \bigwedge_{i<j'} z \not\leq y'_i$

Sketch of Compton's proof: equivalences

Isomorphism lemma

If $\mathcal{M}_1$ and $\mathcal{M}_2$ are countably infinite models with $\mathcal{M}_1 \models T_{po}$ and $\mathcal{M}_2 \models T_{po}$, then $\mathcal{M}_1 \cong \mathcal{M}_2$.

Proof sketch: Back-and-forth construction. First add three unary relations to the models for the levels L_1, L_2, L_3. Map elements to elements at the same level when extending the partial isomorphism.

Let $\mathcal{R}_{po}$ be the unique countably infinite model with $\mathcal{R}_{po} \models T_{po}$.

Theorem: Equivalences

For each formula σ, the following are equivalent:

1. $\mathcal{R}_{po} \models \sigma$
2. $T_{po} \vdash \sigma$
3. $\lim_{n\to\infty} \mu_n(\sigma) = 1$ (on finite partial orders)

The zero-one law for finite partial orders follows.
The proof can be adapted for finite strict (irreflexive) partial orders.

Overview

Reminder: models of modal logics

Definition: Modal language

Let $\Phi = \{p_1, \ldots, p_k\}$ be a finite set of propositional atoms.
$L(\Phi)$, the modal language over Φ, is the smallest set closed under:

1. If $p \in \Phi$, then $p \in L(\Phi)$.
2. If $A \in L(\Phi)$ and $B \in L(\Phi)$, then also $\neg A \in L(\Phi)$, $(A \wedge B) \in L(\Phi)$, $(A \vee B) \in L(\Phi)$, $(A \rightarrow B) \in L(\Phi)$, $\Box A \in L(\Phi)$, $\Diamond(\varphi) \in L(\Phi)$

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Definition: Model $M = (W, R, V)$

- W is a non-empty set of worlds
- R is a binary accessibility relation
- V assigns to each atomic proposition p in each world $w \in W$ a truth value: $V_w(p) = 0$ (false) or $V_w(p) = 1$ (true)

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The truth definition is as usual, including:
$M, w \models \Box\varphi$ iff for all w' such that wRw', $M, w' \models \varphi$
$M, w \models \Diamond\varphi$ iff there is a w' such that wRw' and $M, w' \models \varphi$

Validity of formulas in models of modal logics

Definition: validity of a formula in a model

Formula φ is valid in $M = (W, R, V)$ (notation $M \models \varphi$) iff for all $w \in W$, $M, w \models \varphi$.
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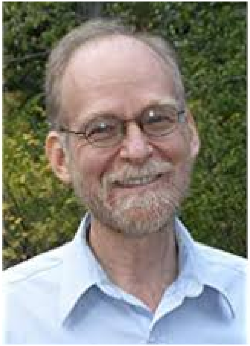
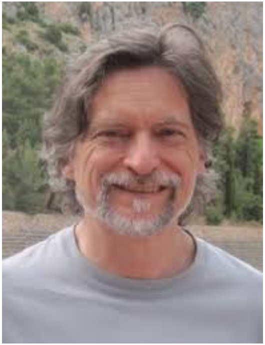
Definition: measure of validity in models of size n

Let $\mathcal{M}_{n,\Phi}$ be the set of finite labeled Kripke models over Φ with set of worlds $W = \{1, \ldots, n\}$.

Let $\nu_{n,\Phi}(\varphi)$ be the measure in $\mathcal{M}_{n,\Phi}$ of the subset of those Kripke models in which φ is valid (based on uniform probability distibution).

Zero-one laws for models of modal logic





Halpern and Kapron (1994) proved that every formula φ in $L(\Phi)$ is either valid in almost all models or not valid in almost all models: Either $\lim_{n\to\infty} \nu_{n,\Phi}(\varphi) = 0$ or $\lim_{n\to\infty} \nu_{n,\Phi}(\varphi) = 1$.

Thus, a zero-one law holds for model validity (no restrictions on the accessibility relation)

What I want to prove

Main aim

I want to investigate whether modal zero-one laws also hold with respect to models of

- provability logic,
- Grzegorczyk logic, and
- weak Grzegorczyk logic.

If so, I want to axiomatize the almost sure validities for each of the corresponding three model classes.

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Secondary aim

What can be said about almost sure *frame* validity in these three (and other) modal logics?

Reminder: A formula φ is *valid in frame* $F = (W, R)$ iff for all valuations V, φ is valid in the model (W, R, V).

Overview

Provability logic GL (Gödel-Löb)

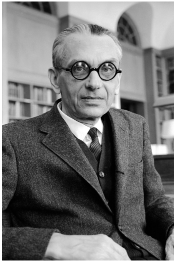
GL contains all axiom schemes from **K** and the extra scheme GL:

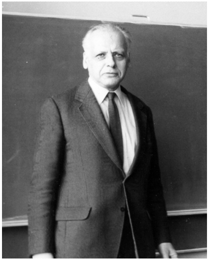
$$\text{All (instances of) propositional tautologies} \tag{A1}$$

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \tag{A2}$$

$$\Box(\Box\varphi \to \varphi) \to \Box\varphi \tag{GL}$$

The rules of inference of **GL** are modus ponens and necessitation.





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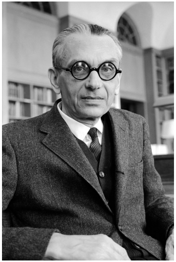
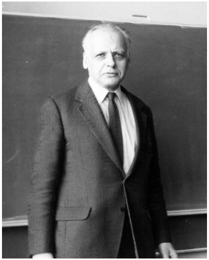
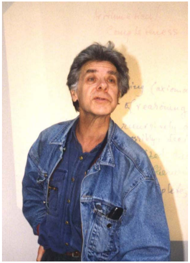


Note that **GL** $\vdash \Box\varphi \to \Box\Box\varphi$ (De Jongh, Sambin, 1973)

Provability logic is sound and complete with respect to all finite, transitive, irreflexive frames (Segerberg, 1971).

Grzegorczyk logic Grz

Grz, a.k.a. **S4Grz**, has the same axiom schemes and inference rules as **GL**, except that axiom GL is replaced by Grz:

$$\Box(\Box(\varphi \to \Box\varphi) \to \varphi) \to \varphi \tag{Grz}$$

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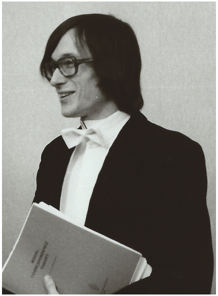


Again, **Grz** $\vdash \Box\varphi \to \Box\Box\varphi$ (Blok and Van Benthem, 1978)
But also **Grz** $\vdash \Box\varphi \to \varphi$
Grz is sound and complete with respect to the class of all finite transitive, reflexive and anti-symmetric frames (Segerberg 1971).

Weak Grzegorczyk logic wGrz

wGrz, a.k.a. **K4Grz**, has the same axiom schemes and inference rules as **GL**, except that axiom GL is replaced by wGrz:

$$\Box^{+}(\Box(\varphi \to \Box\varphi) \to \varphi) \to \varphi \tag{wGrz}$$

Here, $\Box^{+}\psi := \Box\psi \wedge \psi$

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Here, $\Box^+\psi := \Box\psi \wedge \psi$

Again, **wGrz** $\vdash \Box\varphi \to \Box\Box\varphi$ (Litak 2007)

However, **wGrz** $\nvdash \Box\varphi \to \varphi$

wGrz is a proper sublogic of **GL** $\cap$ **Grz** (Litak 2007)

wGrz is sound and complete w.r.t. the class of all finite transitive, anti-symmetric frames (need be neither irreflexive nor reflexive).

Relations between the three logics GL, Grz, wGrz

Goldblatt (1978) a.o. proved that **Grz** can be faithfully and fully translated into **GL**. Define the splitting translation by:

- $p_i^+ = p_i$ for atomic sentences $p_i \in \Phi$;
- $(\varphi \wedge \psi)^+ = (\varphi^+ \wedge \psi^+)$ (similarly other connectives);
- $(\Box\varphi)^+ = \Box\varphi^+ \wedge \varphi^+$.

Then for all $\varphi \in L(\Phi)$: $\mathbf{Grz} \vdash \varphi$ if and only if $\mathbf{GL} \vdash \varphi^+$

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Then for all $\varphi \in L(\Phi)$: **Grz** $\vdash \varphi$ if and only if **GL** $\vdash \varphi^+$

Esakia (2006) proved that the splitting translation $^+$ also faithfully and fully translates **Grz** into **wGrz**:

$$\textbf{Grz} \vdash \varphi \text{ if and only if } \textbf{wGrz} \vdash \varphi^+$$

Overview

GL: Zero-one law for finite irreflexive transitive models

Let $\mathcal{M}_{n,\Phi}$ be the set of finite labeled irreflexive transitive Kripke models over Φ with set of worlds $W = \{1, \ldots, n\}$.
Let $\nu_{n,\Phi}(\varphi)$ be the measure in $\mathcal{M}_{n,\Phi}$ of the subset of those models in which φ is valid.

Theorem (0-1 law)

For every formula φ in $L(\Phi)$:
Either $\lim_{n\to\infty} \nu_{n,\Phi}(\varphi) = 0$ or $\lim_{n\to\infty} \nu_{n,\Phi}(\varphi) = 1$.

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We will show this in two ways:

1. in an easy way by Van Benthem's translation
2. in a more informative way, providing an axiomatization of almost sure validities.

GL: Easy proof of the zero-one law

Van Benthem's translation method (1976 / 1983)

Let * be the translation from $L(\Phi)$ to FOL given by:

- $p_i^* = P_i(x)$ for atomic sentences $p_i \in \Phi$;
- $(\neg\varphi)^* = \neg\varphi^*$;
- $(\varphi \wedge \psi)^* = (\varphi^* \wedge \psi^*)$ (similarly for other binary connectives);
- $(\Box\varphi)^* = \forall y(Rxy \to \varphi^*[y/x])$ (similarly for $\Diamond$).

Van Benthem mapped each Kripke model $M = (W, R, V)$ to a classical model M^* with as objects the worlds in W and the obvious binary relation R, while each $P_i = \{w \in W \mid M, w \models p_i\}$. He proved that $M \models \varphi$ iff $M^* \models \forall x\, \varphi^*$.

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Corollary of Compton's 0-1 law for finite irreflexive orders

Now for each modal formula $\varphi \in L(\Phi)$,
either $\lim_{n\to\infty} \mu_n(\forall x\, \varphi^*) = 1$ or $\lim_{n\to\infty} \mu_n(\forall x\, \varphi^*) = 0$, so
either $\lim_{n\to\infty} \nu_n(\varphi) = 1$ or $\lim_{n\to\infty} \nu_n(\varphi) = 0$

$\mathbf{GL}$: Axiomatizing the almost sure model validities

Axiom system $\mathbf{AX}^{\mathbf{\Phi},\mathbf{M}}_{\mathbf{GL}}$ has the axioms and rules of $\mathbf{GL}$ plus:

$$\Box\Box\Box\bot \tag{T3}$$

$$\Diamond\top \to \Diamond A \tag{C1}$$

$$\Diamond\Diamond\top \to \Diamond(B \land \Diamond C) \tag{C2}$$

In the axiom schemes C1 and C2, the formulas A, B and C all stand for consistent conjunctions of literals over Φ.

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In the axiom schemes C1 and C2, the formulas A, B and C all stand for consistent conjunctions of literals over Φ.

Example

For $\Phi = \{p_1, p_2\}$, the axiom scheme C1 boils down to:

$$\Diamond\top \to \Diamond(p_1 \wedge p_2)$$

$$\Diamond\top \to \Diamond(p_1 \wedge \neg p_2)$$

$$\Diamond\top \to \Diamond(\neg p_1 \wedge p_2)$$

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Note that $\mathbf{AX}^{\Phi,\mathbf{M}}_{\mathbf{GL}}$ is a propositional theory closed under MP, but not closed under uniform substitution, so not a normal modal logic

A canonical asymptotic Kripke model for GL

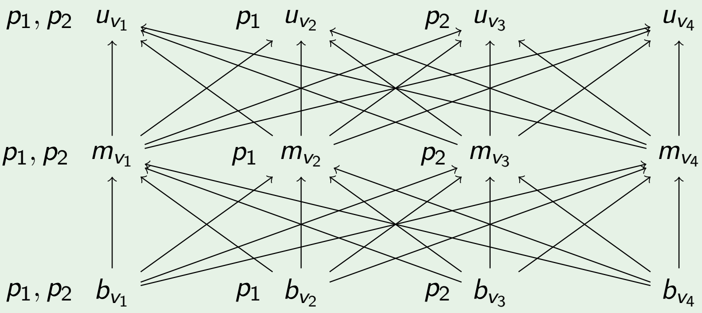
Define $M^{\Phi}_{GL} = (W, R, V)$ as follows:

$W = \{b_v, m_v, u_v \mid v \text{ a propositional valuation on } \Phi\}$;

$R = \{\langle b_v, m_{v'}\rangle \mid v, v' \text{ propositional valuations on } \Phi\} \cup$
$\{\langle m_v, u_{v'}\rangle \mid v, v' \text{ propositional valuations on } \Phi\} \cup$
$\{\langle d_v, u_{v'}\rangle \mid v, v' \text{ propositional valuations on } \Phi\}$; and

for all $p_i \in \Phi$, V is defined by $V_{b_v/m_v/u_v}(p_i) = 1$ iff $v(p_i) = 1$

Example (for $\Phi = \{p_1, p_2\}$)



GL: Zero-one law for finite irreflexive transitive models

The zero-one law for model validity now follows from the theorem:

Theorem

For every formula $\varphi \in L(\Phi)$, the following are equivalent:

1. $M^{\Phi}_{GL} \models \varphi$;
2. $\mathbf{AX}^{\mathbf{\Phi},\mathbf{M}}_{\mathbf{GL}} \vdash \varphi$;
3. $\lim_{n\to\infty} \nu_{n,\Phi}(\varphi) = 1$;
4. $\lim_{n\to\infty} \nu_{n,\Phi}(\varphi) \neq 0$.

The proof is by a circle of implications $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 1$.

$3 \Rightarrow 4$ is trivial. Let's sketch the other steps.

GL: Zero-one law for finite irreflexive transitive models: proof 1 $\Rightarrow$ 2

For every formula $\varphi \in L(\Phi)$, 1 $\Rightarrow$ 2:

1. $M^{\Phi}_{GL} \models \varphi$;
2. $\mathbf{AX}^{\mathbf{\Phi},\mathbf{M}}_{\mathbf{GL}} \vdash \varphi$;

Example (Proof sketch by contraposition, for $\Phi = \{p_1, p_2\}$)

Let $\varphi \in L(\Phi)$ with $\mathbf{AX}^{\mathbf{\Phi},\mathbf{M}}_{\mathbf{GL}} \nvdash \varphi$. Then $\neg\varphi$ is $\mathbf{AX}^{\mathbf{\Phi},\mathbf{M}}_{\mathbf{GL}}$-consistent. By Lindenbaum, extend it to a maximal $\mathbf{AX}^{\mathbf{\Phi},\mathbf{M}}_{\mathbf{GL}}$-consistent set Δ. Define the canonical model $M^{\Phi,M}_{GL} = (W, R, V)$:

- $W = \{w_\Gamma \mid \Gamma$ is maximally $\mathbf{AX}^{\mathbf{\Phi},\mathbf{M}}_{\mathbf{GL}}$-consistent$\}$.
- $R = \{\langle w_\Gamma, w_\Delta \rangle \mid w_\Gamma, w_\Delta \in W$ and for all $\Box\psi \in \Gamma, \psi \in \Delta\}$
- For each $w_\Gamma \in W : V_{w_\Gamma}(p) = 1$ iff $p \in \Gamma$

As usual, $M^{\Phi,M}_{GL}, w_\Delta \not\models \varphi$. The model is isomorphic to the canonical asymptotic model M^{Φ}_{GL}. Therefore, $M^{\Phi}_{GL} \not\models \varphi$.

GL: Zero-one law for models: proof 2 $\Rightarrow$ 3

For every formula $\varphi \in L(\Phi)$, 2 $\Rightarrow$ 3:

2. $\mathbf{AX}^{\mathbf{\Phi},\mathbf{M}}_{\mathbf{GL}} \vdash \varphi$;
3. $\lim_{n\to\infty} \nu_{n,\Phi}(\varphi) = 1$;

Example (Proof sketch: C1 almost surely true, $\Phi = \{p_1, p_2\}$)

To show: $\Diamond\top \to \Diamond(p_1 \wedge \neg p_2)$ holds in almost all K-R models.
Consider a state s in a K-R model of n elements where $\Diamond\top$ holds.
Then s has asymptotically at least $\frac{1}{4} \cdot \frac{1}{2} \cdot n$ direct successors.
The probability that some state t is a direct successor of s that makes $p_1 \wedge \neg p_2$ true is therefore $\geq \frac{1}{8} \cdot \frac{1}{2^2} = \frac{1}{32}$.
Thus, the probability that s does not have *any* direct successors in which $p_1 \wedge \neg p_2$ holds is $\leq (1 - \frac{1}{32})^n$.
Therefore, the probability that there is *some* s in a K-R model not having *any* direct successors satisfying $p_1 \wedge \neg p_2$ is $\leq n \cdot (1 - \frac{1}{32})^n$.
It is known that $\lim_{n\to\infty} n \cdot (1 - \frac{1}{32})^n = 0$, so
$\lim_{n\to\infty} \nu_{n,\Phi}(\Diamond\top \to \Diamond(p_1 \wedge \neg p_2)) = 1$.

GL: Zero-one law for models: $4 \Rightarrow 1$

For every formula $\varphi \in L(\Phi)$, $4 \Rightarrow 1$:

1. $M^{\Phi}_{GL} \models \varphi$;
4. $\lim_{n\to\infty} \nu_{n,\Phi}(\varphi) \neq 0$.

Example (Proof sketch by contraposition, for $\Phi = \{p_1, p_2\}$)

Suppose that $M^{\Phi}_{GL}, s \not\models \varphi$. To show: this counter-model can be copied into almost every K-R model as they grow large enough.
Consider a large K-R model $M' = (W', R', V')$ of three layers.
As example, suppose s is in the middle layer of M^{Φ}_{GL}.
Large enough M' will have an s' in the middle layer with:

- the same valuation for p_1, p_2 as M^{Φ}_{GL}, s, and
- with direct access to at least 4 different states in the top layer of M', for all 4 valuations.

So M^{Φ}_{GL}, s and M', s' satisfy the same formulas.
Similarly for s in top or bottom layer of M.
Conclusion: $\lim_{n\to\infty} \nu_{n,\Phi}(\varphi) = 0$.

Grz: Zero-one law for finite reflexive transitive antisymmetric models

Define axiom system $\mathbf{AX}^{\mathbf{\Phi},\mathbf{M}}_{\mathbf{Grz}}$ as **Grz** plus the following axioms:

$$\varphi \rightarrow \neg\Diamond(\neg\varphi \wedge \psi \wedge \Diamond(\neg\psi \wedge \chi \wedge \Diamond\neg\chi)) \tag{D3}$$

$$(\varphi \wedge \Diamond\neg\varphi) \rightarrow \Diamond A \tag{C3}$$

$$(\varphi \wedge \Diamond(\neg\varphi \wedge \psi \wedge \Diamond\neg\psi)) \rightarrow \Diamond(B \wedge \Diamond C) \tag{C4}$$

In these axiom schemes, φ, ψ, χ stand for any formulas in $L(\Phi)$; A, B, C stand for consistent conjunctions of literals over Φ.

Example ($\Phi = \{p_1, p_2\}$)

The axiom scheme C4 boils down to:

$$(\varphi \wedge \Diamond\neg\varphi) \rightarrow \Diamond(p_1 \wedge p_2)$$

$$(\varphi \wedge \Diamond\neg\varphi) \rightarrow \Diamond(p_1 \wedge \neg p_2)$$

$$(\varphi \wedge \Diamond\neg\varphi) \rightarrow \Diamond(\neg p_1 \wedge p_2)$$

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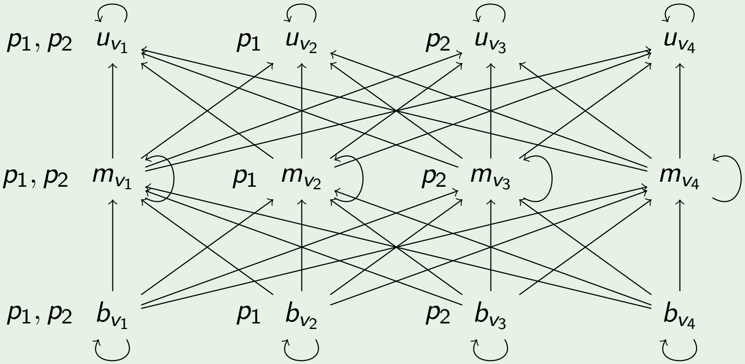
$\mathbf{AX}^{\mathbf{\Phi},\mathbf{M}}_{\mathbf{Grz}}$ is closed under MP, but not under uniform substitution

The canonical asymptotic Kripke model

The canonical asymptotic Kripke model M^{Φ}_{Grz} for **Grz** is the reflexive closure of the one for **GL**.

Example (for $\Phi = \{p_1, p_2\}$)



The 0-1 law for **Grz** can be proved analogously to the one for **GL**.
Note: almost sure model validities for **Grz** and **S4** coincide.

wGrz: Zero-one law for finite transitive antisymmetric models

Define axiom system $\mathbf{AX}^{\mathbf{\Phi},\mathbf{M}}_{\mathbf{Grz}}$ as **wGrz** plus the following axioms:

$$\varphi \to \neg\Diamond(\neg\varphi \wedge \psi \wedge \Diamond(\neg\psi \wedge \chi \wedge \Diamond\neg\chi)) \tag{D3}$$

$$(\varphi \wedge \Diamond\neg\varphi) \to \Diamond A \tag{C3}$$

$$(\varphi \wedge \Diamond(\neg\varphi \wedge \psi \wedge \Diamond\neg\psi)) \to \Diamond(B \wedge \Diamond C) \tag{C4}$$

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In these axiom schemes, φ, ψ, χ stand for any formulas in $L(\Phi)$; A, B, C stand for consistent conjunctions of literals over Φ.

The canonical asymptotic Kripke model M^{Φ}_{wGrz} for **wGrz** is an algamation of those for **GL** and for **Grz**, with a reflexive and an irreflexive copy of each world (corresponding to each propositional valuation at all three levels).

The 0-1 law for **wGrz** can be proved analogously to the one for **Gl**.
Note: almost sure model validities for **wGrz** and **K4** coincide.

Complexity of decision problems for the three siblings

It is known that the satisfiability problems for the modal logics **GL**, **Grz**, and **wGrz** are PSPACE-complete, just like those for **K**, **K4**, and **S4**.

Complexity of almost sure model validity

In contrast, for the three modal logics **GL**, **Grz**, and **wGrz**, if we restrict to language $L(\Phi)$ with finite Φ, the problem of deciding whether $\lim_{n\to\infty} \nu_{n,\Phi}(\varphi) = 1$ is in PTIME: just check the appropriate finite canonical model.

If Φ is enumerably infinite, these problems are in Δ_2^p (Halpern and Kapron, 1994).

Overview

Zero-one laws: An introduction

Zero-one laws for modal logics

The three logics

Zero-one laws over relevant classes of finite models

How about zero-one laws for classes of finite frames?

Conclusions and current work

Reminder: frames of modal logics

Definition: Frame $F = (W, R)$

- W is a non-empty set of worlds
- R is a binary accessibility relation

Definition: validity of a formula in a frame

Formula φ is valid in frame $F = (W, R)$ iff
for all valuations V over $L(\Phi)$ and all $w \in W$, $M, w \models \varphi$

So validity in *all* models in a class coincides with validity in *all* frames in that class.

Frame validity in the limit

Definition: measure of validity in frames of size n

Let $\mathcal{F}_{n,\Phi}$ be the set of finite labeled Kripke frames over Φ with set of worlds $W = \{1, \dots, n\}$.

Let $\mu_{n,\Phi}(\varphi)$ be the measure in $\mathcal{F}_{n,\Phi}$ of the subset of those Kripke frames in which φ is valid (based on uniform probability distribution).

Frame validity in the limit

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Example

Note that almost sure model validity and almost sure frame validity behave quite differently.

For example, $\Diamond\top \to \Diamond p_1$ is valid in almost all **GL**-models,
but not in almost all **GL**-frames:
For every Kleitman-Rothschild frame, take a valuation that makes p_1 false everywhere. Clearly, $\lim_{n\to\infty} \nu_{n,\Phi}(\Diamond\top \to \Diamond p_1) = 0$

Surprising history: 0-1 laws for frame validity for **K**, **T**?

Halpern and Kapron (1994) proposed four axiomatizations for each of the sets of formulas that would be almost surely valid in the four classes of frames corresponding to **K**, **T**, **S4** and **S5**.

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Le Bars (2003) proved Halpern and Kapron wrong for **K**:
There is *no* 0-1 law with respect to **K**-frames.
$q \wedge \neg p \wedge \Box\Box((p \vee q) \to \neg\Diamond(p \vee q)) \wedge \Box\Diamond p$ does *not* have an asymptotic probability. This can probably be adapted for **T**.

Surprising history: 0-1 laws for frame validity for **S4**?

Halpern and Kapron (1994) proposed the following axiomatization for the set of formulas that would be almost surely valid in reflexive transitive frames corresponding to **S4**'.

S4$'$ contains all axiom schemes from **K** and the extra schemes:

$$\Box\varphi \to \Box\Box\varphi \tag{4}$$

$$\Box\varphi \to \varphi \tag{T}$$

$$\neg(p \wedge \Diamond(\neg p \wedge \Diamond(p \wedge \Diamond\neg p))) \tag{DEP2'}$$

The rules of inference of **S4**$'$ are modus ponens and necessitation.

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The rules of inference of **S4**$'$ are modus ponens and necessitation.

But **S4**$'$ is *not* complete for almost sure frame validities (V 2018).

Example (non-completeness of S4$'$)

$(p \wedge q \wedge \Diamond(\neg p \wedge \Diamond p \wedge \Box r)) \to \Box((\neg q \wedge \Diamond q) \to \Diamond r)$
characterizes the diamond property on three levels:
$\forall x \in L_1, \forall y, z \in L_2((Rxy \wedge Rxz) \to \exists u \in L_3(Ryu \wedge Rzu))$.
So it is valid in almost all reflexive Kleitman-Rothschild frames; but it does not follow from **S4**$'$.

How about **GL**?

Take $\Phi = \{p_1, \ldots, p_k\}$. The axiom system $\mathbf{AX}^{\Phi,\mathbf{F}}_{\mathbf{GL}}$ has the same axioms and rules as **GL**, plus the following axiom schemas, for all $k \in \mathbb{N}$, where all $\varphi_i \in L(\Phi)$:

$$\Box\Box\Box\bot \qquad \text{(DEPTH2)}$$

$$\Diamond\Diamond\top \wedge \bigwedge_{i\leq k} \Diamond(\Diamond\top \wedge \Box\varphi_i) \rightarrow \Box(\Diamond\top \rightarrow \Diamond(\bigwedge_{i\leq k} \varphi_i)) \qquad \text{(DIAMOND-k)}$$

$$\Diamond\Diamond\top \wedge \bigwedge_{i\leq k} \Diamond(\Box\bot \wedge \varphi_i) \rightarrow \Diamond(\bigwedge_{i\leq k} \Diamond\varphi_i) \qquad \text{(UMBRELLA-k)}$$

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Example (DIAMOND-0)

DIAMOND-0 is the formula
$\Diamond\Diamond\top \wedge \Diamond(\Diamond\top \wedge \Box\varphi_0) \rightarrow \Box(\Diamond\top \rightarrow \Diamond(\varphi_0))$,
which characterizes the 'diamond' property that if a bottom layer world has two direct successors in the middle layer, then these have a common successor in the top layer.

How about **GL**? (cont.)

Take $\Phi = \{p_1, \ldots, p_k\}$. The axiom system $\mathbf{AX}^{\mathbf{\Phi},\mathbf{F}}_{\mathbf{GL}}$ has the same axioms and rules as **GL**, plus the following axiom schemas, for all $k \in \mathbb{N}$, where all $\varphi_i \in L(\Phi)$:

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Example (UMBRELLA-0)

UMBRELLA-0 is the formula $\Diamond\Diamond\top \wedge \Diamond(\Box\bot \wedge \varphi_0) \rightarrow \Diamond\Diamond\varphi_0$, which characterizes the property that bottom layer worlds don't have any *direct* successor in the top layer, but only via an intermediate world in the middle layer.

GL: Zero-one law for finite irreflexive transitive frames

Note that $\mathbf{AX}^{\mathbf{\Phi},\mathbf{F}}_{\mathbf{GL}} \vdash \varphi$ is not finitely axiomatizable by a result of Tarksi: the DIAMOND-k and UMBRELLA-k sequences have strictly increasing strength.

Conjecture: 0-1 law for GL-frames

For every formula $\varphi \in L(\Phi)$, the following are equivalent:

1. $\mathbf{AX}^{\mathbf{\Phi},\mathbf{F}}_{\mathbf{GL}} \vdash \varphi$;
2. $\lim_{n\to\infty} \mu_{n,\Phi}(\varphi) = 1$;
3. $\lim_{n\to\infty} \mu_{n,\Phi}(\varphi) \neq 0$.

The proof is by a circle of implications $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$.

$2 \Rightarrow 3$ is trivial.
$3 \Rightarrow 1$ is work in progress.
Let's sketch $1 \Rightarrow 2$.

GL: Zero-one law for frames: proof $1 \Rightarrow 2$

For every formula $\varphi \in L(\Phi)$, $1 \Rightarrow 2$:

If $\mathbf{AX}_{\mathbf{GL}}^{\mathbf{\Phi},\mathbf{F}} \vdash \varphi$, then $\lim_{n\to\infty} \mu_{n,\Phi}(\varphi) = 1$

In all finite irreflexive K-R frames, **GL** $+ \Box\Box\Box\bot$ is valid.
We check almost-sure frame validity of DIAMOND-k and UMBRELLA-k.

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Almost sure validity of DIAMOND-k

$\Diamond\Diamond\top \wedge \bigwedge_{i\leq k} \Diamond(\Diamond\top \wedge \Box\varphi_i) \rightarrow \Box(\Diamond\top \rightarrow \Diamond(\bigwedge_{i\leq k} \varphi_i))$
characterizes a k-fold, three layer version of the diamond property:
$\forall w \in L_1 \forall x_0 \ldots x_k \in L_2(\bigwedge_{i\leq k} wRx_i \rightarrow \exists z \in L_3(\bigwedge_{i\leq k} x_i Rz))$.

This property follows from an irreflexive version of Compton's extension axioms, and is therefore almost surely the case in K-R frames.

Therefore, $\Diamond\Diamond\top \wedge \bigwedge_{i\leq k} \Diamond(\Diamond\top \wedge \Box\varphi_i) \rightarrow \Box(\Diamond\top \rightarrow \Diamond(\bigwedge_{i\leq k} \varphi_i))$ is almost surely valid in finite irreflexive transitive frames.

$\mathbf{GL}$: Zero-one law for frames: proof $1 \Rightarrow 2$

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In all finite irreflexive K-R frames, **GL** $+ \Box\Box\Box\bot$ is valid.
We check almost-sure frame validity of DIAMOND-k and UMBRELLA-k.

Almost sure validity of UMBRELLA-k

$\Diamond\Diamond\top \wedge \bigwedge_{i\leq k} \Diamond(\Box\bot \wedge \varphi_i) \to \Diamond(\bigwedge_{i\leq k} \Diamond\varphi_i)$ characterizes a k-fold, three layer 'umbrella' property:
$\forall w \in L_1 \forall x_0 \ldots x_k \in L_3(\bigwedge_{i\leq k} wRx_i \to \exists z \in L_2(wRz \wedge \bigwedge_{i\leq k} zRx_i))$.

This property follows from an irreflexive version of Compton's extension axioms, and is therefore almost surely the case in K-R frames.

Therefore, $\Diamond\Diamond\top \wedge \bigwedge_{i\leq k} \Diamond(\Box\bot \wedge \varphi_i) \to \Diamond(\bigwedge_{i\leq k} \Diamond\varphi_i)$ is almost surely valid in finite irreflexive transitive frames.

Conclusions and current work

Conclusion

Zero-one laws hold for finite models of provability logic, Grzegorczyk logic and weak Grezgorczyk logic.

For all three logics, the almost sure model validities can be axiomatized.

Current work

Finish the proof of completeness of the (infinite) axiomatization of the almost sure frame validities of **GL**

Give the correct (infinite) axiomatizations for almost sure frame validities for **S4**, **Grz**, **K4** and **wGrz**.