

Precipitous Ideals, Generic Ultrapowers, and Destroying Saturation

Joint with Dima Sinapova

Noah Schoem

University of Illinois at Chicago

Logic Colloquium 2019

Context

1. Forcing
2. Class Ultrapowers

Class Ultrapowers

1. $U \subseteq \mathcal{P}^V(A)$ a V -ultrafilter on A , i.e.
 - ▶ U a filter on A
 - ▶ If $B \in V$, $B \subseteq A$, then $B \in U$ or $A \setminus B \in U$
2. $Ult(V, U) = \{[f] \mid f : A \rightarrow V, f \in V\}$ by:
 - $[f] \in^* [g]$ iff $\{a \in A \mid f(a) \in g(a)\} \in U$
 - $[f] = [g]$ iff $\{a \in A \mid f(a) = g(a)\} \in U$
3. $j : V \rightarrow Ult(V, U)$, $j(x) = [a \mapsto x]$ is elementary

Table of Contents

Core Definitions

Foreman's Open Question, and Prior Work

Foreman's Open Question at an Inaccessible

Definition

I ideal on $\mathcal{P}(\kappa)$ is $I \subseteq \mathcal{P}(\kappa)$ with

1. $\emptyset \in I, \kappa \notin I$
2. $A \in I, B \subseteq A \implies B \in I$
3. $A, B \in I \implies A \cup B \in I$

Definition

I ideal on $\mathcal{P}(\kappa)$ is $I \subseteq \mathcal{P}(\kappa)$ with

1. $\emptyset \in I, \kappa \notin I$
2. $A \in I, B \subseteq A \implies B \in I$
3. $A, B \in I \implies A \cup B \in I$

Notation

- ▶ $A \in \check{I}$ if $\kappa \setminus A \in I$
- ▶ $A \in I^+$ if $A \notin I$

Definition

I ideal on $\mathcal{P}(\kappa)$ is $I \subseteq \mathcal{P}(\kappa)$ with

1. $\emptyset \in I, \kappa \notin I$
2. $A \in I, B \subseteq A \implies B \in I$
3. $A, B \in I \implies A \cup B \in I$

Notation

- ▶ $A \in \check{I}$ if $\kappa \setminus A \in I$
- ▶ $A \in I^+$ if $A \notin I$

Further, I is

- ▶ κ -complete if $\tau < \kappa, \langle N_\alpha \mid \alpha < \tau \rangle \subseteq I \implies \bigcup_{\alpha < \tau} N_\alpha \in I$
- ▶ normal if closed under *diagonal unions*:
if $\langle N_\alpha \mid \alpha < \kappa \rangle \subseteq I$, then
 $\nabla_{\alpha < \kappa} N_\alpha = \{\beta < \kappa \mid \exists \alpha < \beta \ \beta \in N_\alpha\} \in I$
equivalently, $f : \kappa \rightarrow \kappa, f(\alpha) < \alpha$ I^+ -often $\implies f$ constant I^+ -often

Example

for $\kappa \geq \omega$: $I_{fin} := \{A \subseteq \kappa \mid |A| < \aleph_0\}$ ideal

Example

for $\kappa \geq \omega$: $I_{fin} := \{A \subseteq \kappa \mid |A| < \aleph_0\}$ ideal

Not ω_1 -complete or normal

Example

for $\kappa \geq \omega$: $I_{fin} := \{A \subseteq \kappa \mid |A| < \aleph_0\}$ ideal

Not ω_1 -complete or normal

Example

for $\kappa \geq \omega_1$: NS_κ κ -complete normal ideal

Definition

I ideal on κ , $A, B \subseteq \kappa$:

$A \leq_I B$ if $A \setminus B \in I$

$A \equiv_I B$ if $A \leq_I B$ and $B \leq_I A$.

Definition

I ideal on κ , $A, B \subseteq \kappa$:

$A \leq_I B$ if $A \setminus B \in I$

$A \equiv_I B$ if $A \leq_I B$ and $B \leq_I A$.

Then $\mathcal{P}(\kappa)/I := \{[A]_{\equiv_I} \mid A \subseteq \kappa\}$, $<_I$ induced by \leq_I .

Definition

I ideal on κ is λ -saturated if
for any $\langle S_\alpha \mid \alpha < \lambda \rangle$ family of I^+ -sets
there are $\alpha < \beta$ such that $S_\alpha \cap S_\beta \in I^+$.

Definition

I ideal on κ is λ -saturated if
for any $\langle S_\alpha \mid \alpha < \lambda \rangle$ family of I^+ -sets
there are $\alpha < \beta$ such that $S_\alpha \cap S_\beta \in I^+$.

Theorem

For I κ -complete and κ -saturated:

$\mathbb{P} := (\mathcal{P}(\kappa)/I, <_I) \setminus [\emptyset]$ is a complete Boolean algebra with κ -cc,
(i.e. no antichains of size $\geq \kappa$)

Similar for κ^+ -saturated.

Definition

I ideal on κ is λ -saturated if

for any $\langle S_\alpha \mid \alpha < \lambda \rangle$ family of I^+ -sets

there are $\alpha < \beta$ such that $S_\alpha \cap S_\beta \in I^+$.

Theorem

For I κ -complete and κ -saturated:

$\mathbb{P} := (\mathcal{P}(\kappa)/I, <_I) \setminus [\emptyset]$ is a complete Boolean algebra with κ -cc,
(i.e. no antichains of size $\geq \kappa$)

Similar for κ^+ -saturated.

Given G \mathbb{P} -generic over V : $U = \{A \in \mathcal{P}^V(\kappa) \mid [A] \in G\}$ is a
 V -ultrafilter that is:

- ▶ V - κ -complete: $\lambda < \kappa$, $\langle B_\alpha \mid \alpha < \lambda \rangle \in V$, each $B_\alpha \in U \implies \bigcap_{\alpha < \lambda} B_\alpha \in U$
- ▶ V -normal: $f : \kappa \rightarrow \kappa$, $f \in V$, $f(\alpha) < \alpha$ U -often $\implies f$ constant U -often

Definition

I ideal on κ is λ -saturated if

for any $\langle S_\alpha \mid \alpha < \lambda \rangle$ family of I^+ -sets

there are $\alpha < \beta$ such that $S_\alpha \cap S_\beta \in I^+$.

Theorem

For I κ -complete and κ -saturated:

$\mathbb{P} := (\mathcal{P}(\kappa)/I, <_I) \setminus [\emptyset]$ is a complete Boolean algebra with κ -cc,
(i.e. no antichains of size $\geq \kappa$)

Similar for κ^+ -saturated.

Given G \mathbb{P} -generic over V : $U = \{A \in \mathcal{P}^V(\kappa) \mid [A] \in G\}$ is a
 V -ultrafilter that is:

- ▶ V - κ -complete: $\lambda < \kappa$, $\langle B_\alpha \mid \alpha < \lambda \rangle \in V$, each $B_\alpha \in U \implies \bigcap_{\alpha < \lambda} B_\alpha \in U$
- ▶ V -normal: $f : \kappa \rightarrow \kappa$, $f \in V$, $f(\alpha) < \alpha$ U -often $\implies f$ constant U -often

$Ult(V, U)$ is well-founded and $j : V \rightarrow Ult(V, U)$, $\text{crit}(j) = \kappa$.

Remark

Often, we write $\mathcal{P}(\kappa)/I$ when we mean the forcing $\mathcal{P}(\kappa)/I \setminus [\emptyset]$.

Weak corollaries of saturation

Definition

For I any ideal on κ :

I *precipitous* if $Ult(V, U)$ well-founded.

Weak corollaries of saturation

Definition

For I any ideal on κ :

I *precipitous* if $Ult(V, U)$ well-founded.

Definition

I is λ -*presaturated* if I is precipitous and $\mathcal{P}(\kappa)/I$ preserves λ .

Theorem

I λ -*presaturated* iff for every $\mu < \lambda$, $\langle A_\alpha \mid \alpha < \mu \rangle$ antichains of $\mathcal{P}(\kappa)/I$,
for densely many $[X]$, for all α , $[X]$ compatible with $< \mu$ -many $[Y] \in A_\alpha$.

Theorem

For κ a regular cardinal, κ^+ -saturated implies κ^+ -presaturated
implies precipitous.

Weak corollaries of saturation

Definition

For I any ideal on κ :

I *precipitous* if $Ult(V, U)$ well-founded.

Definition

I is λ -*presaturated* if I is precipitous and $\mathcal{P}(\kappa)/I$ preserves λ .

Theorem

I λ -*presaturated* iff for every $\mu < \lambda$, $\langle A_\alpha \mid \alpha < \mu \rangle$ antichains of $\mathcal{P}(\kappa)/I$,
for densely many $[X]$, for all α , $[X]$ compatible with $< \mu$ -many $[Y] \in A_\alpha$.

Theorem

For κ a regular cardinal, κ^+ -saturated implies κ^+ -presaturated
implies precipitous.

The converses are false.

Table of Contents

Core Definitions

Foreman's Open Question, and Prior Work

Foreman's Open Question at an Inaccessible

Question (Foreman)

For J ideal on $Z \subseteq \mathcal{P}(\kappa^{+n+1})$,

Let I projected ideal to $Z' \subseteq \mathcal{P}(\kappa^{+n})$, i.e.

$$I = \{N \cap z \cap \kappa^{+n} \mid N \in J, z \in Z\}$$

Suppose canonical homomorphism $\mathcal{P}(Z')/I \rightarrow \mathcal{P}(Z)/J$ regular

Question (Foreman)

For J ideal on $Z \subseteq \mathcal{P}(\kappa^{+n+1})$,

Let I projected ideal to $Z' \subseteq \mathcal{P}(\kappa^{+n})$, i.e.

$$I = \{N \cap z \cap \kappa^{+n} \mid N \in J, z \in Z\}$$

Suppose canonical homomorphism $\mathcal{P}(Z')/I \rightarrow \mathcal{P}(Z)/J$ regular

Then: Is I κ^{+n+1} -saturated?

Question (Foreman)

For J ideal on $Z \subseteq \mathcal{P}(\kappa^{+n+1})$,

Let I projected ideal to $Z' \subseteq \mathcal{P}(\kappa^{+n})$, i.e.

$I = \{N \cap z \cap \kappa^{+n} \mid N \in J, z \in Z\}$

Suppose canonical homomorphism $\mathcal{P}(Z')/I \rightarrow \mathcal{P}(Z)/J$ regular

Then: Is I κ^{+n+1} -saturated?

Theorem (Cox, Eskew, Zeman)

No. For counterexample, suffices to find some I

κ^{+n+1} -presaturated, non- κ^{+n+1} -saturated ideal on κ^{+n} .

Theorem (Cox, Eskew)

Such ideals consistently exist for $n = 0$ and κ successor of singular

Theorem

Same with $n = 0$ and $\kappa = \mu^+$ regular, with $\mu^{<\mu} = \mu$, forcable by a specific poset $\mathbb{P}(\mu, \kappa)$

Question

What about κ inaccessible?

Table of Contents

Core Definitions

Foreman's Open Question, and Prior Work

Foreman's Open Question at an Inaccessible

Theorem (Cox, Eskew 2019)

For μ regular, $\mu^{<\mu} = \mu$, and $\kappa = \mu^+$.

Let $\mathbb{P}(\mu, \kappa)$ force a club through κ with $< \mu$ -conditions.

Then if V has κ -complete, κ^+ -saturated ideals on κ :

Theorem (Cox, Eskew 2019)

For μ regular, $\mu^{<\mu} = \mu$, and $\kappa = \mu^+$.

Let $\mathbb{P}(\mu, \kappa)$ force a club through κ with $< \mu$ -conditions.

Then if V has κ -complete, κ^+ -saturated ideals on κ :

- ▶ $V^{\mathbb{P}(\mu, \kappa)} \models$ no κ^+ -saturated ideals on κ

Theorem (Cox, Eskew 2019)

For μ regular, $\mu^{<\mu} = \mu$, and $\kappa = \mu^+$.

Let $\mathbb{P}(\mu, \kappa)$ force a club through κ with $< \mu$ -conditions.

Then if V has κ -complete, κ^+ -saturated ideals on κ :

- ▶ $V^{\mathbb{P}(\mu, \kappa)} \models$ no κ^+ -saturated ideals on κ
- ▶ For I κ -complete, κ^+ -saturated in V ,
 $V^{\mathbb{P}(\mu, \kappa)} \models \bar{I} = \{A \in \mathcal{P}^{V[G]}(\kappa) \mid \exists N \in I \ A \subseteq N\}$ is
 κ^+ -presaturated

Theorem (Cox, Eskew 2019)

For μ regular, $\mu^{<\mu} = \mu$, and $\kappa = \mu^+$.

Let $\mathbb{P}(\mu, \kappa)$ force a club through κ with $< \mu$ -conditions.

Then if V has κ -complete, κ^+ -saturated ideals on κ :

- ▶ $V^{\mathbb{P}(\mu, \kappa)} \models$ no κ^+ -saturated ideals on κ
- ▶ For I κ -complete, κ^+ -saturated in V ,
 $V^{\mathbb{P}(\mu, \kappa)} \models \bar{I} = \{A \in \mathcal{P}^{V[G]}(\kappa) \mid \exists N \in I \ A \subseteq N\}$ is
 κ^+ -presaturated

Remark

$\mathbb{P}(\omega_1, \omega_2)$: Baumgartner-Taylor (1982), forces club of ω_2 with finite conditions

Theorem (Sinapova, S.)

Let $V \models GCH$, with κ inaccessible, κ -complete κ^+ saturated ideals on κ .

Let \mathbb{Q} be Easton support iteration of $\mathbb{P}(\mu, \mu^+)$, $\mu < \kappa$ regular.

Theorem (Sinapova, S.)

Let $V \models GCH$, with κ inaccessible, κ -complete κ^+ saturated ideals on κ .

Let \mathbb{Q} be Easton support iteration of $\mathbb{P}(\mu, \mu^+)$, $\mu < \kappa$ regular.

Then $V^{\mathbb{Q}} \models$

- ▶ *No κ -complete, κ^+ -saturated ideals on κ*

Theorem (Sinapova, S.)

Let $V \models GCH$, with κ inaccessible, κ -complete κ^+ saturated ideals on κ .

Let \mathbb{Q} be Easton support iteration of $\mathbb{P}(\mu, \mu^+)$, $\mu < \kappa$ regular.
Then $V^{\mathbb{Q}} \models$

- ▶ No κ -complete, κ^+ -saturated ideals on κ
- ▶ If I κ -complete, κ^+ -saturated in V
then $\bar{I} = \{A \in \mathcal{P}^{V^{\mathbb{Q}}}(\kappa) \mid \exists N \in I \ A \subseteq N\}$ is κ^+ -presaturated

Towards a (partial) proof: Key properties of $\mathbb{P}(\mu, \mu^+)$

For μ regular, $\mathbb{P}(\mu, \mu^+)$ has:

1. $|\mathbb{P}(\mu, \mu^+)| = \mu^+$ hence $\mathbb{P}(\mu, \mu^+)$ has the μ^{++} -cc

Towards a (partial) proof: Key properties of $\mathbb{P}(\mu, \mu^+)$

For μ regular, $\mathbb{P}(\mu, \mu^+)$ has:

1. $|\mathbb{P}(\mu, \mu^+)| = \mu^+$ hence $\mathbb{P}(\mu, \mu^+)$ has the μ^{++} -cc
2. $\mathbb{P}(\mu, \mu^+)$ is $< \mu$ -directed closed

Towards a (partial) proof: Key properties of $\mathbb{P}(\mu, \mu^+)$

For μ regular, $\mathbb{P}(\mu, \mu^+)$ has:

1. $|\mathbb{P}(\mu, \mu^+)| = \mu^+$ hence $\mathbb{P}(\mu, \mu^+)$ has the μ^{++} -cc
2. $\mathbb{P}(\mu, \mu^+)$ is $< \mu$ -directed closed
3. $\mathbb{P}(\mu, \mu^+)$ is not μ^+ -cc below any condition

Towards a (partial) proof: Key properties of $\mathbb{P}(\mu, \mu^+)$

For μ regular, $\mathbb{P}(\mu, \mu^+)$ has:

1. $|\mathbb{P}(\mu, \mu^+)| = \mu^+$ hence $\mathbb{P}(\mu, \mu^+)$ has the μ^{++} -cc
2. $\mathbb{P}(\mu, \mu^+)$ is $< \mu$ -directed closed
3. $\mathbb{P}(\mu, \mu^+)$ is not μ^+ -cc below any condition
4. For every $\theta \geq \mu^{++}$ and $M \prec (H_\theta, \in, \mu^+)$ with $M \cap \mu^+ \in \mu^+ \cap \text{cof}(\mu)$, $\mathbb{P}(\mu, \mu^+)$ is strongly proper for M

Towards a (partial) proof: Key properties of $\mathbb{P}(\mu, \mu^+)$

For μ regular, $\mathbb{P}(\mu, \mu^+)$ has:

1. $|\mathbb{P}(\mu, \mu^+)| = \mu^+$ hence $\mathbb{P}(\mu, \mu^+)$ has the μ^{++} -cc
2. $\mathbb{P}(\mu, \mu^+)$ is $< \mu$ -directed closed
3. $\mathbb{P}(\mu, \mu^+)$ is not μ^+ -cc below any condition
4. For every $\theta \geq \mu^{++}$ and $M \prec (H_\theta, \in, \mu^+)$ with $M \cap \mu^+ \in \mu^+ \cap \text{cof}(\mu)$, $\mathbb{P}(\mu, \mu^+)$ is strongly proper for M
5. In $V[G]$, $G \mathbb{P}(\mu, \mu^+)$ -generic, is C_μ a club subset of μ^+ with $X \in V, |X| \geq \mu \implies X \not\subseteq C_\mu$

Towards a (partial) proof: Destroying Saturation

With \mathbb{Q} Easton support iteration of $\mathbb{P}(\mu, \mu^+)$, $\mu < \kappa$ regular:

Towards a (partial) proof: Destroying Saturation

With \mathbb{Q} Easton support iteration of $\mathbb{P}(\mu, \mu^+)$, $\mu < \kappa$ regular:

Proposition

\mathbb{Q} is κ -cc, preserves cardinals, and doesn't alter cofinalities.

Towards a (partial) proof: Destroying Saturation

With \mathbb{Q} Easton support iteration of $\mathbb{P}(\mu, \mu^+)$, $\mu < \kappa$ regular:

Proposition

\mathbb{Q} is κ -cc, preserves cardinals, and doesn't alter cofinalities.

Proposition (Destroy saturation)

If J κ -complete κ^+ saturated in V , then \bar{J} is not κ^+ -saturated in $V^{\mathbb{Q}}$.

Towards a (partial) proof: Destroying Saturation

With \mathbb{Q} Easton support iteration of $\mathbb{P}(\mu, \mu^+)$, $\mu < \kappa$ regular:

Proposition (Destroy saturation)

If J κ -complete κ^+ saturated in V , then \bar{J} is not κ^+ -saturated in $V^{\mathbb{Q}}$.

Towards a (partial) proof: Destroying Saturation

With \mathbb{Q} Easton support iteration of $\mathbb{P}(\mu, \mu^+)$, $\mu < \kappa$ regular:

Proposition (Destroy saturation)

If J κ -complete κ^+ saturated in V , then \bar{J} is not κ^+ -saturated in $V^{\mathbb{Q}}$.

Lemma (Baumgartner-Taylor, Laver)

With $j : V \rightarrow \text{Ult}(V, U)$ generic elementary embedding given by $\mathcal{P}(\kappa)/J$:

\bar{J} is κ^+ -saturated in $V^{\mathbb{Q}}$ iff $\Vdash_{\mathcal{P}(\kappa)/J} \text{"}\dot{j}(\mathbb{Q}) \text{ is } \kappa^+\text{-cc"}$

Towards a (partial) proof: Destroying Saturation

With \mathbb{Q} Easton support iteration of $\mathbb{P}(\mu, \mu^+)$, $\mu < \kappa$ regular:

Proposition (Destroy saturation)

If J κ -complete κ^+ saturated in V , then \bar{J} is not κ^+ -saturated in $V^{\mathbb{Q}}$.

Lemma (Baumgartner-Taylor, Laver)

With $j : V \rightarrow \text{Ult}(V, U)$ generic elementary embedding given by $\mathcal{P}(\kappa)/J$:

\bar{J} is κ^+ -saturated in $V^{\mathbb{Q}}$ iff $\Vdash_{\mathcal{P}(\kappa)/J} \text{"}\dot{j}(\mathbb{Q}) \text{ is } \kappa^+\text{-cc"}$

Proof of proposition.

In $V^{\mathcal{P}(\kappa)/J}$, U a $\mathcal{P}(\kappa)/J$ -generic:

$j(\kappa)$ inaccessible in $\text{Ult}(V, U)$ hence $j(\kappa) > \kappa^+$ in $\text{Ult}(V, U)$.

Towards a (partial) proof: Destroying Saturation

With \mathbb{Q} Easton support iteration of $\mathbb{P}(\mu, \mu^+)$, $\mu < \kappa$ regular:

Proposition (Destroy saturation)

If J κ -complete κ^+ saturated in V , then \bar{J} is not κ^+ -saturated in $V^{\mathbb{Q}}$.

Lemma (Baumgartner-Taylor, Laver)

With $j : V \rightarrow \text{Ult}(V, U)$ generic elementary embedding given by $\mathcal{P}(\kappa)/J$:

\bar{J} is κ^+ -saturated in $V^{\mathbb{Q}}$ iff $\Vdash_{\mathcal{P}(\kappa)/J} \dot{j}(\mathbb{Q})$ is κ^+ -cc"

Proof of proposition.

In $V^{\mathcal{P}(\kappa)/J}$, U a $\mathcal{P}(\kappa)/J$ -generic:

$j(\kappa)$ inaccessible in $\text{Ult}(V, U)$ hence $j(\kappa) > \kappa^+$ in $\text{Ult}(V, U)$.

In $\text{Ult}(V, U)$, $j(\mathbb{Q})(\kappa) = \mathbb{P}(\kappa, \kappa^+)$ hence $j(\mathbb{Q})$ is not κ^+ -cc. □

Towards a (partial) proof: Preserving (some) Presaturation

Proposition

*For $J \in V$ a κ -complete normal κ^+ -saturated ideal:
In $V^{\mathbb{Q}}$, \bar{J} is κ^+ -presaturated.*

Towards a (partial) proof: Preserving (some) Presaturation

Proposition

*For $J \in V$ a κ -complete normal κ^+ -saturated ideal:
In $V^{\mathbb{Q}}$, \bar{J} is κ^+ -presaturated.*

Tools we need:

Lemma

*If $\mathbb{Q}_1, \dots, \mathbb{Q}_n$ posets, such that
 $\Vdash_{\mathbb{Q}_1 * \dots * \mathbb{Q}_i} \mathbb{Q}_{i+1}$ λ -cc or λ -directed closed
then $\mathbb{Q}_1 * \dots * \mathbb{Q}_n$ is λ -presaturated.*

Lemma

*If $\mathbb{P} * \mathbb{Q}$ λ -presaturated
then \mathbb{P} λ -presaturated and $\Vdash_{\mathbb{P}} \mathbb{Q}$ λ -presaturated.*

Towards a (partial) proof: Preserving (some) Presaturation

Lemma (Foreman's Duality Theorem)

For J a κ -complete normal precipitous ideal in V

\mathbb{P} a κ -cc poset in V :

Then

$$\mathcal{B}(\mathbb{P} * \mathcal{P}(\kappa)/\bar{J}) \cong \mathcal{B}(\mathcal{P}(\kappa)/J * \dot{j}(\mathbb{P}))$$

Towards a (partial) proof: Preserving (some) Presaturation

Proposition

*For $J \in V$ a κ -complete normal κ^+ -saturated ideal:
In $V^{\mathbb{Q}}$, \bar{J} is κ^+ -presaturated.*

Proof.

For U a $\mathcal{P}(\kappa)/J$ -generic filter: $\dot{j}(\mathbb{Q}) \cong \mathbb{Q} * \dot{\mathbb{R}}$,
 $\dot{\mathbb{R}}$ Easton support iteration of $\mathbb{P}(\mu, \mu^+)$, over $\mu \in [\kappa^+, j(\kappa))$ regular
in $Ult(V, U)$.

Towards a (partial) proof: Preserving (some) Presaturation

Proposition

*For $J \in V$ a κ -complete normal κ^+ -saturated ideal:
In $V^{\mathbb{Q}}$, \bar{J} is κ^+ -presaturated.*

Proof.

For U a $\mathcal{P}(\kappa)/J$ -generic filter: $\dot{j}(\mathbb{Q}) \cong \mathbb{Q} * \dot{\mathbb{R}}$,
 $\dot{\mathbb{R}}$ Easton support iteration of $\mathbb{P}(\mu, \mu^+)$, over $\mu \in [\kappa^+, j(\kappa))$ regular
in $Ult(V, U)$.

For $\mathcal{P}(\kappa)/J * \mathbb{Q} * \dot{\mathbb{R}}$:
 $\mathcal{P}(\kappa)/J$ is κ^+ -cc, \mathbb{Q} is κ -cc (hence κ^+ -cc), and $\dot{\mathbb{R}}$ is κ^+ -directed
closed.

Towards a (partial) proof: Preserving (some) Presaturation

Proposition

*For $J \in V$ a κ -complete normal κ^+ -saturated ideal:
In $V^{\mathbb{Q}}$, \bar{J} is κ^+ -presaturated.*

Proof.

For U a $\mathcal{P}(\kappa)/J$ -generic filter: $\dot{j}(\mathbb{Q}) \cong \mathbb{Q} * \dot{\mathbb{R}}$,
 $\dot{\mathbb{R}}$ Easton support iteration of $\mathbb{P}(\mu, \mu^+)$, over $\mu \in [\kappa^+, j(\kappa))$ regular
in $Ult(V, U)$.

For $\mathcal{P}(\kappa)/J * \mathbb{Q} * \dot{\mathbb{R}}$:

$\mathcal{P}(\kappa)/J$ is κ^+ -cc, \mathbb{Q} is κ -cc (hence κ^+ -cc), and $\dot{\mathbb{R}}$ is κ^+ -directed
closed.

Thus $\mathcal{P}(\kappa)/J * \dot{j}(\mathbb{Q})$ is κ^+ -presaturated

Towards a (partial) proof: Preserving (some) Presaturation

Proposition

For $J \in V$ a κ -complete normal κ^+ -saturated ideal:
In $V^{\mathbb{Q}}$, \bar{J} is κ^+ -presaturated.

Proof.

For U a $\mathcal{P}(\kappa)/J$ -generic filter: $\dot{j}(\mathbb{Q}) \cong \mathbb{Q} * \dot{\mathbb{R}}$,
 $\dot{\mathbb{R}}$ Easton support iteration of $\mathbb{P}(\mu, \mu^+)$, over $\mu \in [\kappa^+, j(\kappa))$ regular
in $Ult(V, U)$.

For $\mathcal{P}(\kappa)/J * \mathbb{Q} * \dot{\mathbb{R}}$:

$\mathcal{P}(\kappa)/J$ is κ^+ -cc, \mathbb{Q} is κ -cc (hence κ^+ -cc), and $\dot{\mathbb{R}}$ is κ^+ -directed
closed.

Thus $\mathcal{P}(\kappa)/J * \dot{j}(\mathbb{Q})$ is κ^+ -presaturated

By Duality, $\mathbb{Q} * \mathcal{P}(\kappa)/\bar{J}$ is κ^+ -presaturated

Towards a (partial) proof: Preserving (some) Presaturation

Proposition

For $J \in V$ a κ -complete normal κ^+ -saturated ideal:
In $V^{\mathbb{Q}}$, \bar{J} is κ^+ -presaturated.

Proof.

For U a $\mathcal{P}(\kappa)/J$ -generic filter: $\dot{j}(\mathbb{Q}) \cong \mathbb{Q} * \dot{\mathbb{R}}$,
 $\dot{\mathbb{R}}$ Easton support iteration of $\mathbb{P}(\mu, \mu^+)$, over $\mu \in [\kappa^+, j(\kappa))$ regular
in $Ult(V, U)$.

For $\mathcal{P}(\kappa)/J * \mathbb{Q} * \dot{\mathbb{R}}$:

$\mathcal{P}(\kappa)/J$ is κ^+ -cc, \mathbb{Q} is κ -cc (hence κ^+ -cc), and $\dot{\mathbb{R}}$ is κ^+ -directed
closed.

Thus $\mathcal{P}(\kappa)/J * \dot{j}(\mathbb{Q})$ is κ^+ -presaturated

By Duality, $\mathbb{Q} * \mathcal{P}(\kappa)/\bar{J}$ is κ^+ -presaturated

Hence in $V^{\mathbb{Q}}$, \bar{J} is κ^+ -presaturated. □

Question (Cox, Eskew 2019)

Can the above be done at the successor of a singular?

Question

Is the use of $GCH_{<\kappa}$ necessary?

Question

Does \mathbb{Q} preserve κ^+ -presaturation of all κ^+ -presaturated ideals?