Over six decades of the model theory of valued fields

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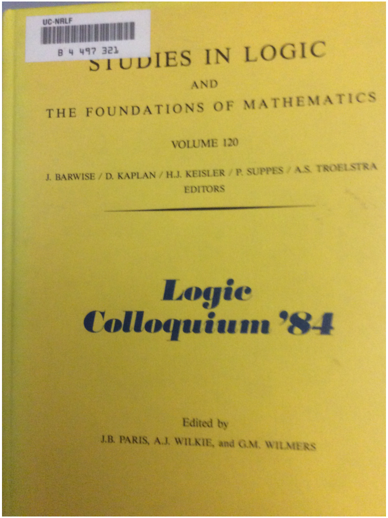
Logic Colloquium 1998

- with J. Krajíček: Combinatorics with definable sets: Euler characteristics and Grothendieck rings, *BSL* **6** , no 3, September 2000, 311 - 330.
- with L. Bélair and A. Macintyre, Model theory of the Frobenius on the Witt vectors, *Amer. J. Math.* **129** Number 3, June 2007, 665 - 721.

Twenty years of *p*-adic model theory from Logic Colloquium 1984



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TWENTY YEARS OF P-ADIC MODEL THEORY

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§0. INTRODUCTION:

I have friends who ridicule the above title, perhaps construing it as mildly pretentious. (Actually I can't remember whether it was I or one of the organizers who suggested it.) At any rate, it makes good sense for me. When I gave my survey lectures in Manchester p-adic model theory was exactly twenty years old, and I had thought about it through most of that time. About half way through those twenty years I had the good luck to make a discovery which is currently regarded as basic (and yet, in the three years following publication of my result, only Kreisel, van den Dries and Denef showed any understanding of it). The phrase "good luck" above is important, not because it confirms my modesty, but because it is meant to prepare the reader for reflections on the somewhat odd and instructive development of p-adic model theory. I will be happy if I succeed in providing an elaborate example for Kreisel's "shifts of emphasis" in logic.

§1. THE CLASSICAL IMPORTANCE OF THE P-ADICS:

1.1. Hensel [Hasse 1980] invented the p-adic numbers. The most obvious use of these objects is to allow free use of rational commutative algebra in the systematic study of congruences. Thus, if $f \in Z[\vec{x}]$ then f has a zero in the p-adic integers if and only if for each k f has a solution modulo p^k. The study of an individual Z/p^k must take account of nilpotent elements, but if one is interested only in the question of solving diophantine equations in all the Z/p^k, then one need only consider solvability in the characteristic 0 domain Z_p (the ring of p-adic integers). The complications due to nilpotents disappear. Moreover, it is entirely natural to replace Z_p by its quotient field Q_p, the field of p-adic numbers.

Hasse [Hasse 1980] points out that Steinitz undertook his fundamental (and model-theoretically influential) treatise on field theory [Steinitz 1930] largely because of these new fields discovered by Hensel.

1.2. As long as one considers only one prime p at a time, one expects to use Z_p (or Q_p) in connection only with a necessary condition for solvability (in Z or Q) of a diophantine equation. But even this can be very powerful. Skolem ([Skolem 1938] or [Borevich-Shafarevich 1966]) used the idea to obtain profound finiteness theorems for certain norm equations. The technique rests on p-adic analytic function theory. Skolem converted the norm equations to p-adic exponential equations, and then employed analytic function theory. It is noteworthy that the method is highly ineffective.

*Supported by N.S.F. Grants
The author takes this chance to record his gratitude to the National Science Foundation of the U.S.A. for very generous support over twelve years at Yale.

Aims of this lecture

- The Ax-Kochen/Ershov theorems and principles have much broader and deeper consequences than previously recognized.
- Robinson's theory of algebraically closed fields has reemerged as a source of profound model theory.
- Continuous logic may offer new insights into the theory of valued fields.

The p-adic numbers from algebraic functions

Hensel was led to the p-adic numbers through his investigations into the theory of algebraic functions and an analogy he observed between such algebraic functions and numbers.

A polynomial function $f : \mathbb{C} \to \mathbb{C}$ is by definition given a finite sum

$$f(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_d t^d$$

where $d \in \mathbb{N}$ and each c_i is a complex number.

More generally, any rational function $f : \mathbb{C} \dashrightarrow \mathbb{C}$, or, indeed, any algebraic function, may be represented by an infinite sum

$$f(t) = c_{-N} t^{-N} + c_{-N+1} t^{-N+1} + \cdots + c_0 + c_1 t + \cdots$$

at least in a neighborhood of the origin.

Considering all such formal series given together with the usual additional and multiplication operations, we obtain the field $\mathbb{C}((t))$ of formal Laurent series.

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The p-adic numbers as Laurent series in p

Fix now a prime number p. Then every natural number may be uniquely expressed in its base-p expansion

$$n = a_0 + a_1 p + a_2 p^2 + \ldots + a_d p^d$$

where each $a_i \in \{0, 1, \ldots, (p-1)\}$. Hensel took the step of representing general rational numbers by possibly infinite Laurent series in p. For example, if $p = 5$, then

$$\frac{3}{20} = \frac{3}{5} + 3 + 3 \cdot 5 + 3 \cdot 25 + \cdots + 3 \cdot 5^d + \cdots$$

The rules for addition and multiplication for this field of p-adic numbers $\mathbb{Q}_p$ may be deduced from the usual rules for adding and multiplying integers taking care to "carry" appropriately.

The p-adic integers, $\mathbb{Z}_p$, consist of those p-adic numbers expressible as $\sum_{n=0}^{\infty} a_n p^n$.

The p-adics as a normed field

There is a natural way to define an absolute value on $\mathbb{Q}_p$: for a non-zero p-adic number expressed as $\alpha = \sum_{n=N}^{\infty} a_n p^n$ where $a_n \in \{0, 1, \ldots, (p-1)\}$, $a_N \neq 0$, and $N \in \mathbb{Z}$, we define $||\alpha||_p := p^{-N} \in \mathbb{R}_+$. (We set $||0||_p := 0$.)

- Restricting $||\cdot||_p$ to $\mathbb{Q}$ regarded as a subfield of $\mathbb{Q}_p$, if $\alpha = p^N \frac{a}{b}$ where a and b are nonzero integers which are coprime to p, then $||\alpha||_p = p^{-N}$.
- For any field k and any choice of real number $\gamma \in (0, 1)$, the field of formal Laurent series $k((t))$, consisting of series $\sum_{n=N}^{\infty} a_n t^n$ with $a_n \in k$ and $N \in \mathbb{Z}$ admits a similar norm defined by $||\sum_{n=N}^{\infty} a_n t^n||_\gamma := \gamma^N$ if $a_N \neq 0$ and $||0|| := 0$. In this way, taking $\gamma := \frac{1}{p}$, the normed field of Laurent series $\mathbb{F}_p((t))$ over the field of p elements looks like the normed field of p-adic numbers $\mathbb{Q}_p$, which is morally "$\mathbb{F}_p((p))$".

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- Restricting $|| \cdot ||_p$ to $\mathbb{Q}$ regarded as a subfield of $\mathbb{Q}_p$, if $\alpha = p^N \frac{a}{b}$ where a and b are nonzero integers which are coprime to p, then $||\alpha||_p = p^{-N}$.
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Anticipating AKE: "interpreting" $\mathbb{Q}_p$ in $\mathbb{F}_p$

The usual construction of the field of Laurent series gives something like an infinitary interpretation of $k((t))$ in k. As such, one might expect that the theory of $k((t))$ is determined by the theory of k.

There is a related construction which starts with a perfect field k of characteristic p and returns a local ring $W(k)$ (the p-typical Witt vectors) realized as $\mathbb{N}$-indexed sequences from k, with maximal ideal $pW(k)$ and $k \cong W(k)/pW(k)$. This construction gives a precise sense in which $\mathbb{Z}_p$ may be realized as "power series in p over $\mathbb{F}_p$". In this sense, we see $\mathbb{Q}_p$ as if it were interpreted in $\mathbb{F}_p$.

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Valued fields

Shortly after Hensel introduced the p-adic numbers as Laurent series in p, the class of valued fields was axiomatized and then $\mathbb{Q}_p$ was realized as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the p-adic norm just as $\mathbb{R}$ may be obtained as the completion of $\mathbb{Q}$ with respect to the usual absolute value.

Already in Kürshák's 1913 Crelle paper "Über Limesbildung und allgemeine Körpertheorie" we find the definition of a valued field as a field K given together with a function $|| \cdot || : K \to \mathbb{R}_{\geq 0}$ satisfying universally:

- $||x|| = 0 \iff x = 0$
- $||xy|| = ||x|| \, ||y||$
- $||1 + x|| \leq 1 + ||x||$

Moreover, the valuation is nontrivial if there is some $a \in K \smallsetminus \{0, 1\}$ with $||a|| \neq 1$ and the valuation is non-archimedian if the triangle inequality may be strengthened to

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Krull (or general) valuations

In the above definition of an absolute value, the values are taken in the nonnegative real numbers, but only the ordered group structure of $\mathbb{R}_+$ is used. To study valued fields from the point of view of first-order logic, it is better to allow for an arbitrary ordered abelian group as the target for the valuation.

A valued field $(K, |\cdot|, \Gamma)$ is a field K given together with a function $|\cdot| : K^\times \to \Gamma$ from K to the ordered abelian group $(\Gamma, \cdot, 1)$ (with the convention that $|0| = 0 < \Gamma$ and $0 \cdot \gamma = \gamma \cdot 0 = 0$ for any $\gamma \in \{0\} \cup \Gamma$) which satisfies universally

- $|xy| = |x|\ |y|$ and
- $|x + y| \leq \max\{|x|, |y|\}$

There are various choices of language in which to express the theory of valued fields and different conventions are followed in the works we discuss. In the simplest one-sorted version, we work with the expansion the language of rings by a binary relation symbol $D(x, y)$ to be interpreted as $D(x, y) :\Longleftrightarrow |x| \leq |y|$. In this language, Γ may be interpreted at $K^\times / \sim$ where $x \sim y :\Longleftrightarrow D(x, y)\ \&\ D(y, x)$.

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Valuation ring, maximal ideal, and residue field

For a valued field $(K, |\cdot|, \Gamma)$, the closed unit ball

$$\mathscr{O}_K := \{x \in K \ : \ |x| \leq 1\}$$

is a subring of K, called its ring of integers, and the open unit ball

$$\mathfrak{m}_K := \{x \in K \ : \ |x| < 1\}$$

is a maximal ideal. The quotient

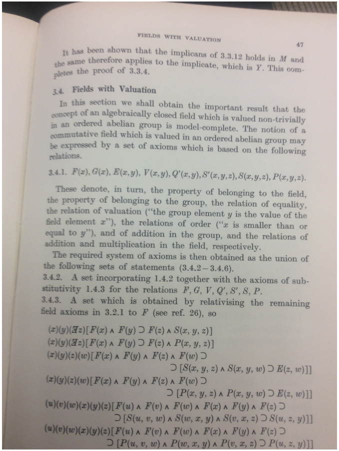
$$k_K = \mathscr{O}_K / \mathfrak{m}_K$$

is the residue field.

We drop the subscript "K" if it is understood.

- In $\mathbb{Q}_p$, $\mathscr{O} =: \mathbb{Z}_p = \{\sum_{n=0}^{\infty} a_n p^n \ : \ a_n \in \{0, \ldots, (p-1)\}\}$ and $k = \mathbb{F}_p$.
- In $F((t))$, $\mathscr{O} = F[[t]] = \{\sum_{n=0}^{\infty} a_n t^n \ : \ a_n \in F\}$, and $k = F$.

Algebraically closed valued fields



FIELDS WITH VALUATION 47

It has been shown that the implicans of 3.3.12 holds in M and the same therefore applies to the implicate, which is Y. This completes the proof of 3.3.4.

3.4. **Fields with Valuation**

In this section we shall obtain the important result that the concept of an algebraically closed field which is valued non-trivially in an ordered abelian group is model-complete. The notion of a commutative field which is valued in an ordered abelian group may be expressed by a set of axioms which is based on the following relations.

3.4.1. $F(x), G(x), E(x,y), V(x,y), Q'(x,y), S'(x,y,z), S(x,y,z), P(x,y,z)$.

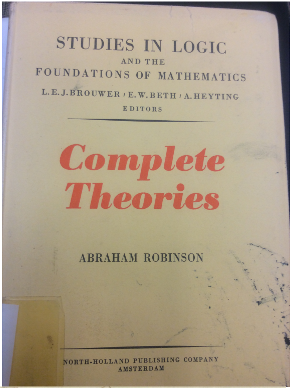
These denote, in turn, the property of belonging to the field, the property of belonging to the group, the relation of equality, the relation of valuation ("the group element y is the value of the field element x"), the relations of order ("x is smaller than or equal to y"), and of addition in the group, and the relations of addition and multiplication in the field, respectively.

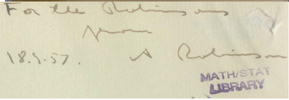
The required system of axioms is then obtained as the union of the following sets of statements (3.4.2 – 3.4.6).

3.4.2. A set incorporating 1.4.2 together with the axioms of substitutivity 1.4.3 for the relations F, G, V, Q', S', S, P.

3.4.3. A set which is obtained by relativising the remaining field axioms in 3.2.1 to F (see ref. 26), so

$$(x)(y)(\exists z)[F(x) \wedge F(y) \supset F(z) \wedge S(x,y,z)]$$
$$(x)(y)(\exists z)[F(x) \wedge F(y) \supset F(z) \wedge P(x,y,z)]$$
$$(x)(y)(z)(w)[F(x) \wedge F(y) \wedge F(z) \wedge F(w) \supset \supset [S(x,y,z) \wedge S(x,y,w) \supset E(z,w)]]$$
$$(x)(y)(z)(w)[F(x) \wedge F(y) \wedge F(z) \wedge F(w) \supset \supset [P(x,y,z) \wedge P(x,y,w) \supset E(z,w)]]$$
$$(u)(v)(w)(x)(y)(z)[F(u) \wedge F(v) \wedge F(w) \wedge F(x) \wedge F(y) \wedge F(z) \supset \supset [S(u,v,w) \wedge S(w,x,y) \wedge S(v,x,z) \supset S(u,z,y)]]$$
$$(u)(v)(w)(x)(y)(z)[F(u) \wedge F(v) \wedge F(w) \wedge F(x) \wedge F(y) \wedge F(z) \supset \supset [P(u,v,w) \wedge P(w,x,y) \wedge P(v,x,z) \supset P(u,z,y)]]$$





Theorem (A. Robinson, 1956)

The theory of non-trivially valued algebraically closed fields, ACVF, is the model companion of the theory of valued fields.

Decidable theories of fields

As of 1963 only three classes of fields were known to be decidable:

- finite fields (individually, for obvious reasons; that the class of finite fields is decidable was shown by Ax only in 1968)
- real closed fields
- algebraically closed fields

and Tarski conjectured that there are no other fields with a decidable theory.

- In a survey on (un)decidable theories composed in 1963 (and published in English in 1965) Yu.L. Ershov, I. A. Lavrov, A.D. Taimanov and M.A. Taitslin raise this and several related questons, such as whether every field of formal Laurent series has an undecidable theory.
- J. Robinson noted that these "fields are decidable because in some sense 'so many' equations are solvable. This suggests that the p-adic fields are promising candidates for counterexamples to Tarski's conjecture."

Hensel's Lemma

Theorem

Let $(K, ||\cdot||)$ be a complete valued field with valuation $||\cdot|| : K^\times \to \mathbb{R}_+$. Let $f(x) \in \mathscr{O}_K[x]$ be a polynomial with integral coefficients. Suppose that $b \in \mathscr{O}_K$ satisfies $||f(b)|| < 1 = ||f'(b)||$. Then there is some $c \in K$ with $f(c) = 0$ and $||b - c|| = ||f(b)||$.

- This lemma applies, in particular, to $\mathbb{Q}_p$ and to fields of formal Laurent series $k((t))$.
- Quantifying over the coefficients of f, Hensel's Lemma may be expressed by a countable set of first-order sentences. A valued field satisfying Hensel's Lemma is called henselian.

Ax-Kochen/Ershov theorem

Theorem (Ax and Kochen; Ershov)

Let $(K, |\cdot|)$ and $(L, |\cdot|)$ be two henselian valued fields whose residue fields have characteristic zero. Then $K \equiv L$ if and only if $k_K \equiv k_L$ and $\Gamma_K \equiv \Gamma_L$.

- It follows that $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$ have the same first-order theory in the limit in the sense that
$$\prod \mathbb{Q}_p/\mathscr{U} \equiv \prod \mathbb{F}_p((t))/\mathscr{U}$$
for any non-principal ultrafilter $\mathscr{U}$ on the set of primes.
- Fixing the prime p, if K and L are two henselian fields of characteristic zero whose residue fields have characteristic p, then $K \equiv L$ if and only if $\Gamma_K \equiv \Gamma_L$ and $\mathscr{O}_K/p^n\mathscr{O}_K \equiv \mathscr{O}_L/p^n\mathscr{O}_L$ for every $n \in \mathbb{Z}_+$.

Definable sets in $\mathbb{Q}_p$

Various quantifier simplification theorems were known early in the work on the model theory of $\mathbb{Q}_p$.

- The Ax-Kochen-Ershov theorem gives quantifier elimination for $\mathbb{Q}_p$ in an expanded language having the cross-section $n \mapsto p^n$ and divisibility predicates on the value group.
- Cohen gave a primitive recursive decision procedure for the theory of $\mathbb{Q}_p$ based on a cell decomposition theorem, still using the cross section $n \mapsto p^n$.
- Macintyre proved quantifier elimination for $\mathbb{Q}_p$ in a language having predicates P_n defined by $P_n(x) :\Longleftrightarrow (\exists y) y^n = x$.

From quantifier elimination to computation of integrals

For a set $S \subseteq \mathbb{Z}_p^m$ and $n \in \mathbb{N}$, define $N_n(S) := \#(S \mod p^n)$ and consider the Poincaré series

$$P_S(T) := \sum_{n=0}^{\infty} N_n(S) T^n$$

When S is definable, then $P_S(T)$ is a rational function of T.

In a paper published in 1984, Denef proves this theorem by relating the rationality if $P_S(T)$ to properties of integrals $\int_{\mathbb{Z}_p^m} |f|_p^s$ where f is a definable function and then uses quantifier elimination (and further properties) to reduce the computation of the integrals to elementary series computations.

The first-order formulae describing S would make sense with p replaced by another prime. Answering the question of how the resulting rational function varies with p motivated the search for uniform quantifier elimination theorems.

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AKE as a principle: relative quantifier elimination

The Ax-Kochen-Ershov principle asserts that properties of the henselian field K may be understood from the corresponding properties of its value group and residue rings (or possibly from more general auxiliary interpretable structues such as $\mathrm{RV}(K) = K^\times/(1+\mathfrak{m})$ or the "geometric sorts").

For example, in the language with a cross-section, henselian fields of residue characteristic zero have quantifier elimination relative to the value group and the residue field k.

Uniform p-adic integration to motivic integration

- Bounds on the degrees of the rational functions appearing in Denef's theorem follow from uniform quantifier elimination theorems for the p-adics. (Macintyre, *APAL* (1990) and Pas, *JLMS* (1990))
- The uniformity is better expressed motivically: in the Poincaré series $P_S(T) = \sum \#(S \mod p^n) T^n$ rather than counting $(S \mod p^n)$ regard the formula $(S \mod p^n)$ as a "number" in its own right as an element of an appropriate Grothendieck ring. Denef and Loeser show in "Definable sets, motives, and p-adic integrals", *JAMS* (2000), that the rationality theorem holds motivically.
- The class of integrals to which theory of Denef-Loeser-Cluckers motivic integration applies has expanded over the years including the kinds of oscillating orbital integrals appearing in the Langland's Program.

Robinson's ACVF returns

- In "Integration in valued fields" (2006), Hrushovski and Kazhdan develop a theory of motivic integration based on the theory $\mathrm{ACVF}_{0,0}$.
- In the 2006 Crelle paper "Definable sets in algebraically closed valued fields: elimination of imaginaries" followed by the 2008 Lecture Notes in Logic "Stable domination and independence in algebraically closed valued fields", Haskell, Hrushovski, and Macperson describe the interpretable sets in ACVF by developing a theory of "stable domination".
- In the 2016 Annals of Mathematics Studies monograph "Non-archimedean tame topology and stably dominated types" Hrushovski and Loeser construct a theory of the geometry of nonarchimedean spaces using spaces of stably dominated types.

Stability for valued fields?

- Theories of (non-trivially) valued fields can never be stable because they have an interpretable order (the value group).
- The AKE principle does hold for other stability-like properties. For instance, a henselian field of residue characteristic zero has the independence property if and only if either its residue field or its value group does.
- The theory of stable domination suggests that the theory ACVF behaves as if it were composed of stable parts parameterized by the value group. This observation has been made precise through the theory of metastability.
- Ben Yaacov shows that if ACVF is considered in continuous logic, then it is stable. This move to continuous logic is the first necessary step in the (still developing) Ben Yaacov-Hrushovski theory of fields with a product formula.

Perfectoid spaces

- The analogy between valued fields of characteristic p and those of mixed characteristic has been given a precise and powerful formulation in Scholze's theory of perfectoid spaces.
- Behind this theory is the tilt/untilt correspondence: given a complete mixed characteristic $(0, p)$ valued field K whose value group is a dense subgroup of $\mathbb{R}_+$ and for which the residue ring $\mathscr{O}/p\mathscr{O}$ is perfect, the tilt $K^\flat$, is a complete valued field of characteristic p constructed from an inverse limit procedure. Using the Witt vector construction followed by a appropriate quotient, from a perfect, complete, nontrivially valued field L of characteristic p, one constructs the untilt $L^\sharp$ in such a way that $(K^\flat)^\sharp \cong K$.
- The correspondence transfers properties between K and $K^\flat$. For example, their absolute Galois groups are isomorphic.

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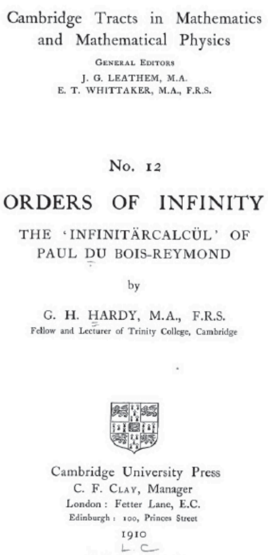
What else?

Some researchers of the model theory of valued fields

Meghan Anderson, Sylvy Anscombe, Matthias Aschenbrenner, James Ax, Şerban Basarab, Jochen Becker, Luc Bélair, Itaï Ben Yaacov, Anna Blaszczok, Ali Bleybel, Zoé Chatzidakis, Gregory Cherlin, Artem Chernikov, Raf Cluckers, Georges Comte, Pablo Cubides Kovacsics, Paola D'Aquino, Luck Darnière, Françoise Delon, Jan Denef, Jamshid Derakhshan, Philip Dittmann, Jan Dobrowolski, Alf Dolich, Antoine Ducros, Salih Durham, Clifton Ealy, Philip Ehrlich, Kirsten Eisenträger, Yuri Ershov, Arno Fehm, Joseph Flenner, Arthur Forey, Antongiulio Fornasiero, Tim Gardener, Allen Gehret, Julia Gordan, Nicholas Guzy, Thomas Hales, Yatir Halevi, Immi Halupczok, Deirdre Haskell, Nadja Hempel, Martin Hils, Joris van der Hoeven, Jan Holly, Ehud Hrushovski, Franziska Jahnka, Moshe Jarden, Will Johnson, Moshe Kamensky, Itay Kaplan, David Kazhdan, Julia Knight, Piotr Kowalski, Jochen Koenigsmann, Simon Kochen, Krzysztof Krupiński, Franz-Viktor Kuhlmann, Salma Kuhlmann, Tristan Kuijpers, Karen Lange, Chris Laskowski, Jung-uk Lee, Wan Lee, Eva Leenknegt, Gérard Leloup, Leonard Lipshitz, François Loeser, Fares Maalouf, Angus Macintyre, Dugald Macpherson, Anatoly Malcev, Jana Mařiková, Nathanaël Mariaule, Ben Martin, Christian Michaux, Samaria Montenegro, Kien Huu Nguyen, Johannes Nicaise, Krzysztof Jan Nowak, Gönenç Onay, Alf Onshuus, Koushik Pal, Jennifer Park, Johan Pas, Hector Paster, Thanases Pheidas, Anand Pillay, Jérôme Poineau, Françoise Point, Florian Pop, Alexander Prestel, Serge Randriambololona, Silvain Rideau, Romain Rioux, Abraham Robinson, Edmund Robinson, Julia Robinson, Raphael Robinson, Zach Robinson, Peter Roquette, Thomas Scanlon, Joachim Schmid, Hans Schoutens, Philip Scowcroft, Saharon Shelah, Sasha Shlapentokh, Pierre Simon, Sergei Starchenko, Tigran Takobyan, Marcus Tressl, Xavier Vidaux, Frank Wagner, Volker Weisspfenning, Yoav Yaffe, Jinhe Ye, Yimu Yin, Martin Ziegler

Aside: Orders of infinity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and Mathematical Physics

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No. 12

ORDERS OF INFINITY

THE 'INFINITÄRCALCÜL' OF
PAUL DU BOIS-REYMOND

by

G. H. HARDY, M.A., F.R.S.
Fellow and Lecturer of Trinity College, Cambridge

Cambridge University Press
C. F. CLAY, Manager
London: Fetter Lane, E.C.
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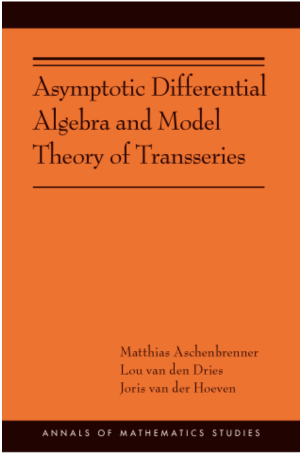
1910



Rychlík's Lemma

Zur Bewertungstheorie der algebraischen Körper.

Von Herrn *Karl Rychlik* in Prag.

Einleitung.

§ 1. Die Körper und Integritätsbereiche sind Bereiche, in denen nach den aus den Elementen der Algebra bekannten Regeln für die rationalen bzw. ganzen Zahlen gerechnet wird [1]).

Man kann auch den Begriff des absoluten Betrages verallgemeinern und auf Grund desselben den Limes und die Konvergenz definieren.

Wir werden jedem Elemente a des Körpers K eine reelle Zahl $||a|| \geqq 0$ zuordnen, so daß folgende Bedingungen erfüllt sind:

I. $||0|| = 0$, *für* $a \neq 0$ *ist* $||a|| > 0$, und wenn der triviale Fall $||0|| = 0$, $||a|| = 1$ für $a \neq 0$, ausgeschlossen werden soll: *Es gibt im Körper wenigstens ein Element* a, *für welches* $||a|| \neq 1$.

II. $||ab|| = ||a|| \cdot ||b||$.

III. $||1 + a|| \leqq 1 + ||a||$.

$||a||$ wird nach Herrn *Kürschák Bewertung* des Elementes a genannt, und ein Körper, für den eine solche Zuordnung existiert, wird als *bewerteter Körper* bezeichnet [2]).

Dann gilt: 1.) $\left\|\frac{a}{b}\right\| = \frac{||a||}{||b||}$, für $b \neq 0$; 2.) $||1|| = 1$; 3.) $||-1|| = 1$;
4.) $||-a|| = ||a||$; 5.) $||a + b|| \leqq ||a|| + ||b||$;
6.) $||a + b + c + \cdots|| \leqq ||a|| + ||b|| + ||c|| + \cdots$.

Im speziellen Falle, daß statt III die Bedingung III′ gilt: *Für* $||a|| \leqq 1$ *ist* $||1 + a|| \leqq 1$, erhalten wir *die nichtarchimedische Bewertung* [3]).

Anstatt (5.) und (6.) gilt dann: 5′.) $||a + b|| \leqq \mathrm{Max}\,(||a||, ||b||)$, und wenn $||a|| \neq ||b||$, sogar $||a + b|| = \mathrm{Max}\,(||a||, ||b||)$.

$$6'.) \quad ||a + b + c + \cdots|| \leqq \mathrm{Max}\,(||a||, ||b||, ||c||, \ldots).$$

Daraus folgt in diesem Falle gleich $||n|| \leqq 1$, wenn n eine ganze rationale Zahl bezeichnet.

---

[1]) *Steinitz*, Dieses Journal 137, 1910. Die Ergebnisse dieser Arbeit setze ich voraus und benutze die dort eingeführte Terminologie.

[2]) *Kürschák*, Dieses Journal 142, 1913, § 1—23.

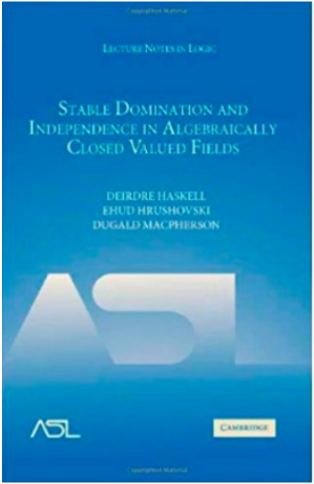
[3]) *Ostrowski*, Acta mathem. 41, 1917.

- Hensel proved Hensel's Lemma for $\mathbb{Q}_p$ and Rychlík extended the proof to general complete valued fields.
- Rychlík's proof originally appeared in Czech in 1917 and then again in German in Crelle in 1924.
- Rychlík's formulation of Hensel's Lemma differs from what we have written.

Robinson's ACVF returns: Hrushovski-Kazhdan motivic integration

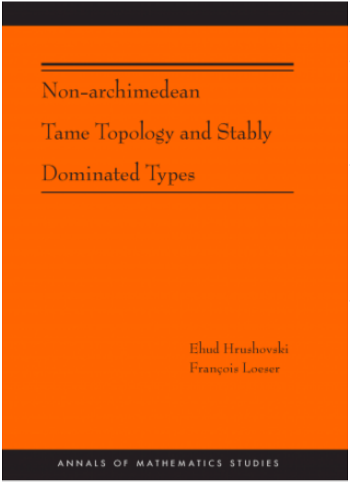
- In motivic integration in the style of Kontsevich-Denef-Loeser-Cluckers, the underlying valued field is always discretely valued so that the definable sets used for counting are always internal to the residue field.
- Using quantifier elimination for ACVF (and a deceptively simple averaging technique that is only valid for residue characteristic zero), Hrushovski and Kazhdan develop an alternate theory of motivic integration in which the sets used for counting are definable sets in powers of RV, and thus take into account both the residue field and the value group.
- Using quantifier elimination for ACVF again, they show how to specialize this theory to the usual theory of motivic integration.

Robinson's ACVF returns: Stable domination and imaginaries



- In general, an imaginary is an element of a quotient X/E where X is definable and E is a definable equivalence relation.
- In valued fields, there are interpretable sets which are not definably isomorphic to definable sets (*e.g.* $\Gamma = K^\times/\mathscr{O}^\times$)
- For ACVF, all such imaginaries are reducible to "geometric imaginaries", from quotients $\mathrm{GL}_n(K)/\mathrm{GL}_n(\mathscr{O})$ and $\mathrm{GL}_n(K)/G_n^0$ where $G_n^0 = \ker(\mathrm{GL}_n(\mathscr{O}) \to \mathrm{GL}_n(k))$.
- The proof depends on developing a sense in which stability theory at the level of the residue field lifts to this unstable theory. (Recall that a theory is unstable if there is a formula $\phi(x,y)$ and in some model sequences $(a_i)_{i=0}^\infty$ and $(b_j)_{j=0}^\infty$ with $\phi(a_i, b_j) \iff i \leq j$.)

Robinson's ACVF returns: Stably dominated types and Berkovich geometry



- Nonarchimedean geometry violates many of our intuitions from Euclidean geometry. In particular, the topology is totally disconnected.
- Various enriched spaces have been proposed, and this model theoretic approach takes the stably dominated types as the points of the geometric space.
- Using the (pro-)definability of these spaces, they may be analyzed through first-order logic and it is shown, for instance, that these spaces admit spaces definable in the value group as deformation retracts.

Ax-Kochen and Ershov

DIOPHANTINE PROBLEMS OVER LOCAL FIELDS I.*

By James Ax and Simon Kochen.

0. Introduction. A conjecture of Artin states that every form f of degree d in $n > d^2$ variables over Q_p, the p-adic completion of the rationals, has a non-trivial zero in Q_p. For the case $d = 2$, this is a classical theorem about quadratic forms. A proof of the conjecture for $d = 3$ was given by Lewis in [13].

In this paper we prove:

(1) *For every positive integer d there exists a finite set of primes $A = A(d)$ such that for every prime $p \notin A$ every form f of degree d in $n > d^2$ variables over Q_p has a non-trivial zero in Q_p.*[1]

This and the analogous assertion for the completion of a number field k (here A depends only on d and $[k:Q]$) follow from Theorem 5. A further result obtained is the following:

(2) *Let f be a polynomial without constant term of degree d in $n > d$ variables over the ring Z of rational integers. Then there exists a finite set B of primes such that for every prime $p \notin B$, f has a non-trivial zero in Q_p.*

(See the Corollary to Theorem 1.) This re-proves[2] a conjecture of Lang [12].

These and similar results are special cases of the following general

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[1] Recently D. J. Lewis and B. J. Birch have proved the special cases $d = 5, 7, 11$ of (1).

[2] After completing the original manuscript, we were informed by D. Lewis that N. Greenleaf has obtained the Corollary to Theorem 1, using algebraic-geometric techniques.

А Л Г Е Б Р А и Л О Г И К А
СЕМИНАР

Том 6
Выпуск 3 РУКОВОДИТЕЛЬ А.И.МАЛЬЦЕВ 1967 г.

ОБ ЭЛЕМЕНТАРНОЙ ТЕОРИИ МАКСИМАЛЬНЫХ НОРМИРОВАННЫХ ПОЛЕЙ. Ш

Ю.Л.ЕРШОВ

Эта работа тесно связана с предыдущей [2] . Цель работы расширить результаты работы [2] в случае неравнохарактеристических полей.

Пусть $\mathcal{F}$ – класс всех полей характеристики $p \neq 0$, $\mathcal{F}_0 \subset \mathcal{F}$ – класс всех совершенных полей этой характеристики . $\mathcal{G}$ – класс всех линейно упорядоченных групп Γ таких, что Γ имеет наименьший положительный элемент 1 , другими словами, в Γ имеется выпуклая подгруппа Z , изоморфная группе целых чисел. Будем рассматривать нормированные поля (нормирования) $\langle F, v, \Gamma \rangle$, удовлетворяющие условию:

$$F \text{ имеет характеристику } 0, \Gamma \in \mathcal{G}, v(p) \in Z \subseteq \Gamma, F_v \in \mathcal{F}; \text{ если } F_v \in \mathcal{F} \setminus \mathcal{F}_0, \text{ то } v(p) = 1. \qquad (*)$$

Если нормирование $\langle F, v, \Gamma \rangle$ удовлетворяет условию $(*)$, то будем говорить, что оно алгебраически полно, если нормирование v – гензелево (см. [2]).

Сформулируем теперь основную теорему.

ТЕОРЕМА. П у с т ь $\mathcal{F}_1 \subseteq \mathcal{F}$ – а к с и о м а т и з и р у е м ы й к л а с с п о л е й, σ_1 – ф о р м у л ь н а я с и г н а т у р а, в к о т о р о й $\mathcal{F}_1$ м о д е л ь н о п о л о н, $\mathcal{G}_1 \subseteq \mathcal{G}$ – а к с и о м а т и з и р у е м ы й к л а с с л и н е й н о у п о р я д о ч е н н ы х а б е л е в ы х г р у п п,